

COBORDISM CATEGORY OF MANIFOLDS WITH BAAS-SULLIVAN SINGULARITIES, PART I

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ABSTRACT. For a fixed closed manifold P , we construct a cobordism category of embedded manifolds with a single Baas-Sullivan singularity of type P . Our main theorem identifies the homotopy type of the classifying space of this cobordism category with that of the infinite loop-space of a certain spectrum related to the spectrum $\mathbf{MT}(d)$ introduced in [6]. We obtain an analogue of the Bockstein-Sullivan exact couple that arises between the classical bordism theories MO and MO_P on the level of cobordism categories.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

Let P be a fixed closed manifold. Let W be a $(d+1)$ -dimensional manifold with corners modeled on $[0, \infty)^2 \times \mathbb{R}^{d-1}$. The boundary of W then decomposes into the union of two manifolds $\partial_0 W$ and $\partial_1 W$, with the property that

$$\partial(\partial_0 W) = \partial_0 W \cap \partial_1 W = \partial(\partial_1 W).$$

Suppose further that there is a factorization, $\partial_1 W = \beta_1 W \times P$ where $\beta_1 W$ is a manifold with boundary. We refer to such manifolds as P -manifolds. The submanifold $\partial_0 W$ is referred to as the “boundary” of the P -manifold W ; closed P -manifolds are P -manifolds with empty ∂_0 component. Denoting by $C(P)$ the cone over P , one can easily attach the product $\beta_1 W \times C(P)$ to $\partial_1 W$ to obtain a manifold with *Baas-Sullivan singularity of type P* . For details on such manifolds, see [1] and [4].

We are interested in the cobordism theory of P -manifolds. To simplify our presentation we will assume that all manifolds are unoriented. We denote by Ω_* the graded cobordism group of unoriented manifolds. The corresponding graded cobordism group Ω_*^P of P -manifolds is closely related to Ω_* by means of the well-known Bockstein-Sullivan exact couple:

$$(1) \quad \begin{array}{ccc} \Omega_* & \xleftarrow{\quad \times P \quad} & \Omega_* \\ & \searrow i & \nearrow \beta_1 \\ & \Omega_*^P & \end{array}$$

which arises from the cofibre sequence of a map of the classifying spectra,

$$\Sigma^P MO \xrightarrow{\quad \times P \quad} MO \longrightarrow MO_P .$$

This construction can be carried out for manifolds with arbitrary tangential structure. Details on the construction of this exact couple can be found in [4]. An important example is the case when $P = \{1, \dots, k\}$. Manifolds of this type are sometimes referred to as \mathbb{Z}/k -manifolds.

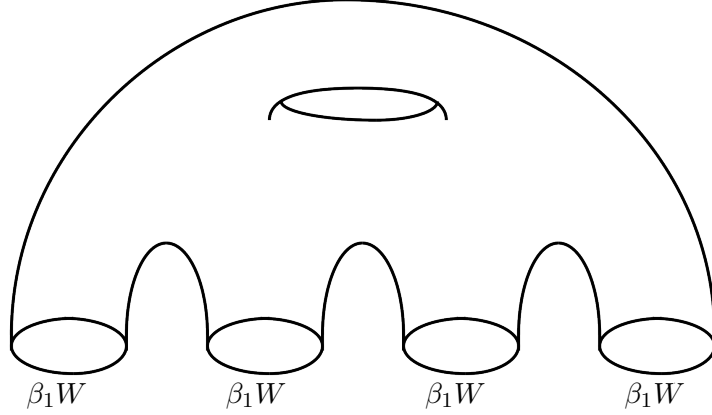


FIGURE 1. A $\mathbb{Z}/4$ -Manifold with empty ∂_0 -component. The boundary decomposes into the disjoint union of 4 identical copies of the same manifold $\beta_1 W$.

Motivated by the ideas in [2], we seek to construct a cobordism category of manifolds with Baas-Sullivan singularities and determine the homotopy-type of its classifying space. In [6], the authors construct a cobordism category \mathbf{Cob}_{d+1} whose morphisms are $(d+1)$ -dimensional submanifolds $W \subseteq [a, b] \times \mathbb{R}^{d+n}$ that intersect the walls $\{a, b\} \times \mathbb{R}^{d+n}$ transversely in ∂W . This category is made into a topological category and there are homotopy equivalences,

$$\mathcal{Ob}(\mathbf{Cob}_{d+1}) \sim \bigsqcup_M \mathrm{BDiff}(M), \quad \mathcal{Mor}(\mathbf{Cob}_{d+1}) = \bigsqcup_W \mathrm{BDiff}(W; \partial_{in}, \partial_{out})$$

where M varies over diffeomorphism classes of d -dimensional manifolds without boundary and W varies over diffeomorphism classes of cobordisms. Here $\mathrm{Diff}(M)$ is the group of diffeomorphisms of M and $\mathrm{Diff}(W; \partial_{in}, \partial_{out})$ is the group of diffeomorphisms of W that restrict to diffeomorphisms of the incoming and outgoing boundaries. In [6], the authors determine the homotopy type of the classifying space of \mathbf{Cob}_{d+1} . They prove that there is a weak homotopy equivalence

$$\mathbf{BCob}_{d+1} \sim \Omega^{\infty-1} \mathbf{MT}(d+1).$$

Here $\mathbf{MT}(d+1)$ is a spectrum whose $(n+d+1)$ th space is the Thom-space $\mathrm{Th}(U_{d+1,n}^\perp)$ where $U_{d+1,n}^\perp$ is the orthogonal complement to the canonical $(d+1)$ -plane bundle over the Grassmanian manifold $G(d+1, n)$ of $(d+1)$ -dimensional vector subspaces of \mathbb{R}^{d+1+n} .

Following this work on cobordism categories, we construct a cobordism category of P -manifolds \mathbf{Cob}_{d+1}^P whose morphisms are $(d+1)$ -dimensional P -submanifolds

$$W \subseteq [a, b] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n}$$

(with possibly non-empty ∂_0 -component), such that

$$W \cap \{a, b\} \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n} = \partial_0 W \quad \text{and} \quad W \cap [a, b] \times \{0\} \times \mathbb{R}^{d-1+n} = \partial_1 W,$$

where both intersections are assumed to be transverse (here, and throughout this paper we have, $\mathbb{R}_+ := [0, \infty)$). We topologize this category in such a way similar to as in [6] so that there are homotopy equivalences,

$$\mathcal{O}b(\mathbf{Cob}_{d+1}^P) \sim \bigsqcup_M \text{BDiff}(M)^{\langle P \rangle}, \quad \mathcal{M}or(\mathbf{Cob}_{d+1}^P) = \bigsqcup_W \text{BDiff}(W)^{\langle P \rangle},$$

where M varies over diffeomorphism classes of d -dimensional P -manifolds with empty ∂_0 -component and W varies over diffeomorphism classes of $(d+1)$ -dimensional P -manifolds with possibly non-empty ∂_0 -component. For a P -manifold W , $\text{Diff}(W)^{\langle P \rangle}$ is defined to be the group of diffeomorphisms $g : W \rightarrow W$ such that the restriction $g|_{\partial_1 W}$ is equal to the product $g_{\beta_1 W} \times Id_P$ where $g_{\beta_1 W}$ is a diffeomorphism of $\beta_1 W$. With this definition, such a diffeomorphism g will extend to a homeomorphism of the singular manifold

$$W \cup_{\partial_1 W} (\beta_1 W \times C(P))$$

that is smooth away from the singular conic-point.

In the special case that the manifold P is the empty-set, the category $\mathbf{Cob}_{d+1}^\emptyset$ is isomorphic to the category \mathbf{Cob}_{d+1} from [6]. In the special case that P is a single point (i.e., $P = \star$), then \mathbf{Cob}_{d+1}^\star is the category whose objects are d -dimensional submanifolds with boundary and whose morphisms are $(d+1)$ -dimensional submanifolds with corners. This case is covered in [7]. Furthermore, for an arbitrary p -dimensional closed manifold P , there is an inclusion functor

$$i : \mathbf{Cob}_{d+1}^P \hookrightarrow \mathbf{Cob}_{d+1}^\star,$$

since by definition a P -manifold is a manifold with corners. There is also a functor,

$$\beta_1 : \mathbf{Cob}_{d+1}^P \longrightarrow \mathbf{Cob}_{d-p}$$

which sends a P -manifold W to the $(d-p)$ -dimensional manifold $\beta_1 W$. It can be checked that these functors are indeed continuous. These continuous functors fit into a pull-back diagram of topological categories,

$$(2) \quad \begin{array}{ccc} \mathbf{Cob}_{d+1}^P & \xrightarrow{i} & \mathbf{Cob}_{d+1}^\star \\ \downarrow \beta_1 & & \downarrow \beta_1 \\ \mathbf{Cob}_{d-p} & \xrightarrow{\times P} & \mathbf{Cob}_d, \end{array}$$

where the bottom-horizontal functor is given by sending a manifold W to $W \times P$. Since the functor $\mathbf{B} : \mathcal{CAT} \rightarrow \mathcal{TOP}$ sending a small category to its classifying space preserves finite limits, this pull-back diagram descends to a pull-back on the level of classifying spaces as well.

One of the goals of this paper is to determine the homotopy type of the classifying space \mathbf{BCob}_{d+1}^P . To do so we construct a new spectrum $\mathbf{MT}(d+1)_{\langle P, \partial \rangle}$ as follows.

Fixing an embedding

$$i_P : P \hookrightarrow \mathbb{R}^{p+m}$$

(where p is the dimension of P), we get a Thom-Pontyagin map $c_P : S^{p+m} \longrightarrow \mathrm{Th}(U_{p,m}^\perp)$. The natural multiplication map

$$\begin{array}{ccc} U_{d-p,n-m}^\perp \times U_{p,m}^\perp & \xrightarrow{\hat{\mu}} & U_{d,n}^\perp \\ \downarrow & & \downarrow \\ G(d-p, n-m) \times G(p, m) & \xrightarrow{\mu} & G(d, n), \end{array}$$

yields a map of Thom-spaces

$$\mathrm{Th}(\hat{\mu}) : \mathrm{Th}(U_{d-p,n-m}^\perp) \wedge \mathrm{Th}(U_{p,m}^\perp) \longrightarrow \mathrm{Th}(U_{d,n}^\perp).$$

The composition

$$\mathrm{Th}(U_{d-p,n-m}^\perp) \wedge S^{p+m} \xrightarrow{c_P \wedge \mathrm{Id}} \mathrm{Th}(U_{d-p,n-m}^\perp) \wedge \mathrm{Th}(U_{p,m}^\perp) \xrightarrow{\mathrm{Th}(\hat{\mu})} \mathrm{Th}(U_{d,n}^\perp)$$

then induces a map of spectra

$$\tau_P : \mathbf{MT}(d-p) \longrightarrow \mathbf{MT}(d).$$

There is another map of spectra $\tilde{j}_d : \Sigma^{-1}\mathbf{MT}(d) \longrightarrow \mathbf{MT}(d+1)$, induced by the bundle map $U_{d,n}^\perp \rightarrow U_{d+1,n}^\perp$ covering the standard embedding $G(d, n) \hookrightarrow G(d+1, n)$. We define $\mathbf{MT}(d+1)_{\langle P, \partial \rangle}$ to be the cofibre of the composition,

$$(3) \quad \Sigma^{-1}\mathbf{MT}(d-p) \xrightarrow{\Sigma^{-1}\tau_P} \Sigma^{-1}\mathbf{MT}(d) \xrightarrow{\tilde{j}_d} \mathbf{MT}(d+1).$$

Details of this construction are covered in Section 6. We now state our main result:

Theorem 1.1 (Main Theorem). *There is a weak homotopy equivalence*

$$\mathbf{BCob}_{d+1}^P \sim \Omega^{\infty-1}\mathbf{MT}(d+1)_{\langle P, \partial \rangle}.$$

Now, the spectrum $\mathbf{MT}(d+1)_{\langle P, \partial \rangle}$ was constructed using the Thom-Pontryagin map for a particular embedding of our manifold P . The homotopy class of this Thom-Pontryagin map only depends on the cobordism-class of P . From this observation we get,

Corollary 1.2. *Let P_1 and P_2 be smooth closed manifolds of the same dimension. Suppose that P_1 and P_2 are cobordant. Then there is a weak homotopy equivalence,*

$$\mathbf{BCob}_{d+1}^{P_1} \sim \mathbf{BCob}_{d+1}^{P_2}.$$

One can consider the functors,

$$(4) \quad \mathbf{Cob}_{d+1}^P \xrightarrow{i} \mathbf{Cob}_{d+1}^P \xrightarrow{\beta_1} \mathbf{Cob}_{d-p}$$

where i is given by inclusion and β_1 sends a $(d+1)$ -dimensional P -manifold W to the $(d-p)$ -dimensional manifold $\beta_1 W$. Applying the classifying space functor $\mathbf{B}(\cdot)$ to (4) we get

$$\mathbf{BCob}_{d+1} \xrightarrow{\mathbf{B}(i)} \mathbf{BCob}_{d+1}^P \xrightarrow{\mathbf{B}(\beta_1)} \mathbf{BCob}_{d-p}.$$

Now since the spectrum $\mathbf{MT}(d+1)_{\langle P, \partial \rangle}$ is defined to be the cofibre of a map

$$\Sigma^{-1}\mathbf{MT}(d-p) \longrightarrow \mathbf{MT}(d+1),$$

there is a homotopy fibre sequence,

$$\Omega^{\infty-1}\mathbf{MT}(d+1) \longrightarrow \Omega^{\infty-1}\mathbf{MT}(d+1)_{\langle P, \partial \rangle} \longrightarrow \Omega^{\infty-1}\mathbf{MT}(d-p).$$

In Section 9 we show that the homotopy-fibre of the map

$$\mathbf{B}(\beta_1) : \mathbf{BCob}_{d+1}^P \longrightarrow \mathbf{BCob}_{d-p}$$

is weakly homotopy equivalent to the space $\Omega^{\infty-1}\mathbf{MT}(d+1)$. Using the fact that

$$\Omega^{\infty-1}\mathbf{MT}(d+1) \sim \mathbf{BCob}_{d+1},$$

we have:

Theorem 1.3. *The homotopy fibre of $\mathbf{B}(\beta_1) : \mathbf{BCob}_{d+1}^P \longrightarrow \mathbf{BCob}_{d-p}$ is weakly equivalent to \mathbf{BCob}_{d+1} .*

This paper is structured as follows. Sections 2 through 4 are devoted to the construction of the category \mathbf{Cob}_{d+1}^P . We give a definition of P -manifolds in Section 2 and in Section 3 we describe the moduli-space of P -manifolds which will enable us to topologize the category \mathbf{Cob}_{d+1}^P in Section 4. In Section 5 we define a sheaf \mathbf{D}_{d+1}^P whose representing space will later be seen to be weakly equivalent to \mathbf{BCob}_{d+1}^P . In Section 6 we construct the spectrum $\mathbf{MT}(d+1)_{\langle P, \partial \rangle}$ and Section 7 is devoted to proving the weak homotopy equivalence of the representing space of \mathbf{D}_{d+1}^P with the infinite loop space $\Omega^{\infty-1}\mathbf{MT}(d+1)_{\langle P, \partial \rangle}$. In Section 9, we show that the representing space of the sheaf \mathbf{D}_{d+1}^P is weakly equivalent to \mathbf{BCob}_{d+1}^P . Sections 10, 11 and the appendices are devoted to the proofs of technical results used earlier in the paper. Our constructions will utilize the concordance theory of sheaves. Concepts such as concordance, representing space, and cocycle-sheaves will be used. For definitions of these mathematical entities we refer the reader to [6] and [11].

In this paper we will only treat unoriented manifolds with a single Baas-Sullivan singularity of type P . One could easily adapt our proofs to derive a corresponding theorem for P -manifolds with arbitrary tangential structure. One could also consider manifolds with multiple nested singularities (see [4]). The general case of cobordism categories of such manifolds will be treated in the sequel.

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2. MANIFOLDS WITH SINGULARITY

We begin with a definition of manifolds with a single Baas-Sullivan singularity. For what follows let P be a closed manifold of dimension p .

Definition 2.1. Let W be a d -dimensional smooth manifold with corners modeled on the space $[0, \infty)^2 \times \mathbb{R}^{d-2}$. We call W a P -manifold if W satisfies the following conditions:

- i. There is a decomposition

$$\partial W = \partial_0 W \cup \partial_1 W$$

of the boundary ∂W into a union of $(d-1)$ -dimensional manifolds with boundary such that the intersection

$$\partial_{0,1} W := \partial_0 W \cap \partial_1 W$$

is a closed d -dimensional manifold. It is also required that

$$\partial(\partial_0 W) = \partial(\partial_1 W) = \partial_{0,1} W.$$

- ii. There are compatible product structures given by diffeomorphisms,

$$\phi_1 : \partial_1 W \xrightarrow{\cong} \beta_1 W \times P,$$

$$\phi_{0,1} : \partial_{0,1} W \xrightarrow{\cong} \beta_{0,1} W \times P,$$

where $\beta_0 W$ and $\beta_{0,1} W$ are both manifolds and $\partial(\beta_1 W) = \beta_{0,1} W$. It is also required that if $i : \partial_{0,1} W \rightarrow \partial_1 W$ is the inclusion map, then the composition

$$\phi_{0,1}^{-1} \circ i \circ \phi_1 : \beta_{0,1} W \times P \rightarrow \beta_1 W \times P$$

takes the form $i_{\beta_{0,1} W} \times Id_P$ where $i_{\beta_{0,1} W}$ is the inclusion of $\beta_{0,1} W$ into $\beta_1 W$. It is convenient to denote $\beta_0 W := \partial_0 W$.

- iii. It is required that our P -manifolds be equipped with embeddings of collar neighborhoods. The embeddings are denoted:

$$h_1 : \partial_1 W \times [0, 1) \rightarrow W,$$

$$h_0 : \partial_0 W \times [0, 1) \rightarrow W.$$

Furthermore, these embeddings have to satisfy the following compatibility conditions:

$$h_0(\partial_{0,1} W \times [0, 1)) \subset \partial_1 W,$$

$$h_1(\partial_{0,1} W \times [0, 1)) \subset \partial_0 W;$$

and for $(w, s, t) \in \partial_{0,1} W \times [0, 1) \times [0, 1)$, the following compositions must be equal:

$$(w, s, t) \longmapsto (h_0(w, s), t) \longmapsto h_1(h_0(w, s), t),$$

$$(w, s, t) \longmapsto (h_1(w, t), s) \longmapsto h_0(h_1(w, t), s).$$

To get actual singularities we do the following. Two points x, y of a P -manifold W are equivalent if they belong to $\partial_1 W$ or $\partial_{0,1} W$ and

$$\begin{aligned} pr \circ \phi_1(x) &= pr \circ \phi_1(y), \\ pr \circ \phi_{0,1}(x) &= pr \circ \phi_{0,1}(y), \end{aligned}$$

where pr is the projection onto the factor of $\beta_1 W$ or $\beta_{0,1} W$. We can then take the quotient with respect to this equivalence relation. This construction is equivalent to attaching the cone over P as described in the introduction.

It will be convenient to deal with P -manifolds prior to taking the quotient by this equivalence relation. We will have to make sure that all maps defined on these P -manifolds respect the product structures described above. In more detail, it is required that for maps

$$f : W \longrightarrow X$$

where W is a P -manifold and X is any space, the restrictions $f|_{\partial_1 W}$, $f|_{\partial_{0,1} W}$ must have the following factorizations

$$(5) \quad \begin{aligned} \partial_1 W &\xrightarrow[\cong]{\phi_1} \beta_1 W \times P^1 \xrightarrow{proj} \beta_1 W \xrightarrow{f_{\beta_1 W}} X, \\ \partial_{0,1} W &\xrightarrow[\cong]{\phi_{0,1}} \beta_{0,1} W \times P \xrightarrow{proj} \beta_{0,1} W \xrightarrow{f_{\beta_{0,1} W}} X, \end{aligned}$$

where $f_{\beta_1 W}$ and $f_{\beta_{0,1} W}$ are continuous maps. Such a map f will always descend to a map on the quotient by the relation defined above.

We are interested in the cobordism theory of P -manifolds. For this we make the following definition.

Definition 2.2. Two d -dimensional P -manifolds M_a and M_b are said to be cobordant if there is a P -manifold W of dimension $d + 1$ such that

$$\partial_0 W \cong M_a \sqcup M_b.$$

In that case we denote

$$h_0|_{M_a} := h_0^a \quad \text{and} \quad h_0|_{M_b} := h_0^b$$

where h_0 is the collar embedding given in the definition of a P -manifold. We will be using this notation in the next section. For this reason, we refer to $\partial_0 W$ as the “boundary” of the P -manifold W , even though this is only one face of the entire boundary ∂W . Notice that $\partial_0 M_a = \emptyset = \partial_0 M_b$.

3. MODULI SPACE OF P -MANIFOLDS

3.1. Diffeomorphisms and Embeddings of P -Manifolds. We may now define the space of diffeomorphisms of a P -manifold W . Recall the definition of the product structure maps $\phi_{0,1}, \phi_1$ and the collar embeddings h_1, h_0 used in the definition of a P -manifold.

Definition 3.1. Let W be a P -manifold with $\partial_0 W = M_a \sqcup M_b$ (either component may be empty). For positive constants ϵ_0, ϵ_1 we define $\text{Diff}(W)_{\epsilon_0, \epsilon_1}^{\langle P \rangle}$ to be the space of diffeomorphisms $f : W \rightarrow W$ that satisfy the following conditions:

- i. The map f is a diffeomorphism of a manifold with corners. The following equalities hold,

$$f(\partial_0 W) = \partial_0 W, \quad f(\partial_1 W) = \partial_1 W, \quad \text{and} \quad f(\partial_{0,1} W) = \partial_{0,1} W.$$

- ii. The restrictions of f to the boundary components have the following factorizations

$$\begin{aligned} f|_{\partial_1 W} &= \phi_1^{-1} \circ (f_{\beta_1 W} \times \text{Id}_P) \circ \phi_1, \\ f|_{\partial_{0,1} W} &= \phi_{0,1}^{-1} \circ (f_{\beta_{0,1} W} \times \text{Id}_P) \circ \phi_{0,1}, \end{aligned}$$

where $f_{\beta_1 W}$ and $f_{\beta_{0,1} W}$ are diffeomorphisms of $\beta_1 W$ and $\beta_{0,1} W$ respectively.

- iii. The map f respects collars of width ϵ_0 and ϵ_1 in the following sense:

$$\begin{aligned} f \circ h_1(w, t) &= h_1(f_1(w), t) \quad \text{for } w \in \partial_1 W, \quad t \in [0, \epsilon_1), \\ f \circ h_0^a(w, t) &= h_0^a(f_0^a(w), t) \quad \text{for } w \in M_a, \quad t \in [0, \epsilon_0), \\ f \circ h_0^b(w, t) &= h_0^b(f_0^b(w), t) \quad \text{for } w \in M_b, \quad t \in (1 - \epsilon_0, 1], \end{aligned}$$

where $h_0^a := h_0|_{M_a}$, $h_0^b := h_0|_{M_b}$, and the maps f_1, f_0^a, f_0^b are diffeomorphisms of $\partial_1 W$, M_a , and M_b respectively.

To eliminate dependence on the fixed values ϵ_0 and ϵ_1 , we take a direct limit,

$$(6) \quad \text{Diff}(W)^{\langle P \rangle} := \text{colim}_{\epsilon_0, \epsilon_1 \rightarrow 0} \text{Diff}(W)_{\epsilon_0, \epsilon_1}^{\langle P \rangle}$$

We now define the space of embeddings of a P -manifold.

Definition 3.2. Let W be a $d + 1$ -dimensional P -manifold with $\partial_0 W = M_a \sqcup M_b$. Let $i_P : P \rightarrow \mathbb{R}^{p+m}$ be an embedding. For positive constants ϵ_0, ϵ_1 we define

$$\text{Emb}(W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m})_{\epsilon_0, \epsilon_1}^{\langle P, m \rangle}$$

to be the space of embeddings $g : W \hookrightarrow [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m}$ subject to the following conditions:

- i. The embedding g is a map of manifolds with corner:

$$\begin{aligned} g(M_a) &\subset \{0\} \times \{0\} \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m}, \\ g(M_b) &\subset \{1\} \times \{0\} \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m}, \\ g(\partial_1 W) &\subset [0, 1] \times \{0\} \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m}, \\ g(\partial_{0,1} W) &\subset \{0, 1\} \times \{0\} \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m}. \end{aligned}$$

- ii. The restrictions of the embedding g to the boundary components of W have the following factorizations,

$$\begin{aligned} g|_{\partial_1 W} &= (g_{\beta_1 W} \times i_P) \circ \phi_1, \\ g|_{\partial_{0,1} W} &= (g_{\beta_{0,1} W} \times i_P) \circ \phi_{0,1}, \end{aligned}$$

where $g_{\beta_1 W}$ and $g_{\beta_{0,1} W}$ are embeddings of $\beta_1 W$ and $\beta_{0,1} W$ into $[0, 1] \times \{0\} \times \mathbb{R}^{d-1+n}$ and $\{0, 1\} \times \{0\} \times \mathbb{R}^{d-1+n}$ respectively.

- iii. The embedding g interacts with the collars in the following way:

$$g \circ h_1(w, t) = (g|_{\partial_1 W}(w), t) \quad \text{for } t \in [0, \epsilon_1)$$

where the second coordinate on the right-hand side of the equality is understood to come from the \mathbb{R}_+ -factor in $[0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m}$ up to permutation of coordinates. Also,

$$\begin{aligned} g \circ h_0^a(w, t) &= (g|_{M_a}(w), t) \quad \text{for } t \in [0, \epsilon_0), \\ g \circ h_0^b(w, t) &= (g|_{M_b}(w), t) \quad \text{for } t \in (1 - \epsilon_0, 1], \end{aligned}$$

where the second coordinate on the right-hand side of the above equalities is understood to come from the $[0, 1]$ factor in $[0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m}$ up to permutation of coordinates.

To eliminate the dependence on n, ϵ_0 , and ϵ_1 we take a direct limit,

$$(7) \quad \text{Emb}(W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m})^{\langle P, m \rangle} = \varinjlim_{\substack{n \rightarrow \infty \\ \epsilon_0, \epsilon_1 \rightarrow 0}} \text{Emb}(W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m})^{\langle P, m \rangle}_{\epsilon_0, \epsilon_1}.$$

On certain occasions we will have to consider smooth maps

$$f : W \longrightarrow [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m}$$

that satisfy the three conditions of Definition 3.2 but which may fail to be an embedding. That is, functions f that respect the corners, collars, and whose restriction $f|_{\partial_1 W}$ have the factorization $f|_{\partial_1 W} = f_{\beta_1 W} \times i_P$ for some smooth function $f_{\beta_1 W} : \beta_1 W \longrightarrow [0, 1] \times \mathbb{R}^{d-1+n}$. We denote the space of such functions by

$$(8) \quad C^\infty(W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m})^{\langle P, m \rangle}.$$

Notice that an element of $C^\infty(W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m})^{\langle P, m \rangle}$ is not the same type of map introduced in the previous section in (5). Both types of maps will be important to consider.

All mapping spaces introduced in this section are topologized using the C^∞ -topology. It is easy to check that

$$\text{Emb}(W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m})^{\langle P, m \rangle} \subset C^\infty(W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m})^{\langle P \rangle}$$

is an open subset.

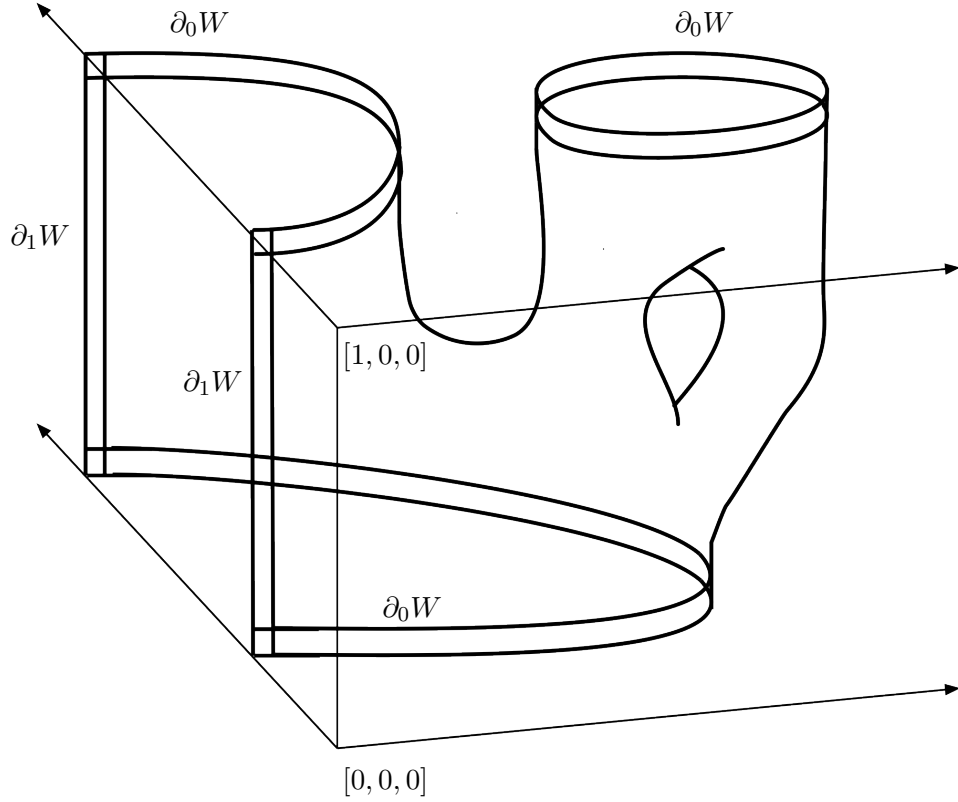


FIGURE 2. A neatly embedded manifold with corners in $[a, b] \times \mathbb{R}_+ \times \mathbb{R}$ with collars.

Remark 3.1. The superscript P in $\text{Emb}(W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{p+m})^{(P,m)}$, refers to a specific embedded submanifold $i_P : P \hookrightarrow \mathbb{R}^{p+m}$. The structure of the space

$$\text{Emb}(W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{p+m})^{(P,m)}$$

depends on this choice. However, the following theorem shows that the homotopy type of this space is independent of the choice of embedding of P .

Theorem 3.1. *Let W be P -manifold. The space $\text{Emb}(W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{p+m})^{(P,m)}$ is weakly contractible.*

Proof. Let $\text{Emb}(W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty})_{\epsilon_0, \epsilon_1}^{(*)}$ be the space of embeddings g of W with

$$\begin{aligned} g(\partial_1 W) &\subseteq [0, 1] \times \{0\} \times \mathbb{R}^{d-1+\infty}, \\ g(\partial_0 W) &\subseteq \{0, 1\} \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty} \end{aligned}$$

respecting collars of length ϵ_0 and ϵ_1 as in Definition 3.2, see Fig 2 above. These embeddings when restricted to the boundary components may not respect the factorization of $\partial_1 W \times P$ as

in Definition 3.2 (ii). This space is equivalent to the space of neat embeddings of a manifold with corners. We then define

$$\text{Emb}(W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty})^{\langle \star \rangle} := \text{colim}_{\epsilon_0, \epsilon_1 \rightarrow \infty} \text{Emb}(W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty})_{\epsilon_0, \epsilon_1}^{\langle \star \rangle}.$$

The space $\text{Emb}(W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{p+m})^{\langle P, m \rangle}$ can be realized as the pull-back of the diagram:

$$(9) \quad \begin{array}{ccc} & & \text{Emb}(W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty})^{\langle \star \rangle} \\ & & \downarrow r \\ \text{Emb}(\beta_1 W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty})^{\langle \star \rangle} & \xrightarrow{pr^*} & \text{Emb}(\partial_1 W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty})^{\langle \star \rangle} \end{array}$$

where the right-hand vertical map r is the restriction map and the bottom-horizontal map is induced by the projection

$$\partial_1 W \cong \beta_1 W \times P \longrightarrow \beta_1 W.$$

Now, it is known that the spaces

$$\begin{aligned} & \text{Emb}(\beta_1 W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty})^{\langle \star \rangle}, \\ & \text{Emb}(N_{\epsilon_1}(\partial_1 W), [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty})^{\langle \star \rangle}, \\ & \text{Emb}(W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty})^{\langle \star \rangle}, \end{aligned}$$

are all weakly contractible; these are simply spaces of neat embeddings of manifolds with boundaries and corners. For proof see [7, Theorem 2.7].

Lemma 3.2. *The restriction map*

$$\text{Emb}(W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty})^{\langle \star \rangle} \xrightarrow{r_{\epsilon_1}} \text{Emb}(\partial_1 W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty})^{\langle \star \rangle}$$

is a Serre fibration.

Proof. We provide a proof of this lemma in Appendix A. □

Clearly Lemma 3.2 implies that

$$\text{Emb}(W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{p+m})^{\langle P, m \rangle}$$

is homotopy equivalent to the homotopy-pullback of the diagram (9). This together with weak-contractibility of the spaces in diagram (9) implies that

$$\text{Emb}(W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{p+m})^{\langle P, m \rangle}$$

is weakly contractible. □

3.2. Fibre Bundles With P -Manifold Fibres. The topological group $\text{Diff}(W)^{\langle P \rangle}$ acts on the space $\text{Emb}(W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{p+m})^{\langle P, m \rangle}$ by pre-composition of an embedding with a diffeomorphism. It is easy to see that this action is a free action. This action induces a quotient map,

$$q : \text{Emb}(W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{p+m})^{\langle P, m \rangle} \longrightarrow \frac{\text{Emb}(W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{p+m})^{\langle P, m \rangle}}{\text{Diff}(W)^{\langle P \rangle}}.$$

Theorem 3.3. *The above quotient map is a principal $\text{Diff}(W)^{\langle P \rangle}$ fibre-bundle.*

Proof. Proof is provided in Appendix A. □

Since both $\text{Emb}(W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{p+m})^{\langle P, m \rangle}$ and $\text{Diff}(W)^{\langle P \rangle}$ are smooth (infinite dimensional) manifolds, the base space of the above principal fibre-bundle is given a canonical smooth structure via the local trivializations. We denote,

$$(10) \quad B_\infty(W)^{\langle P, m \rangle} := \frac{\text{Emb}(W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{p+m})^{\langle P, m \rangle}}{\text{Diff}(W)^{\langle P \rangle}},$$

and will often refer to $B_\infty(W)^{\langle P, m \rangle}$ as the moduli-space of the P -manifolds diffeomorphic to W . Notice that as a set, $B_\infty(W)^{\langle P, m \rangle}$ is exactly the set of all P -submanifolds of

$$[0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{p+m}$$

that are diffeomorphic to W . The weak contractibility of

$$\text{Emb}(W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{p+m})^{\langle P, m \rangle}$$

along with the fibre-bundle structure implies that there is a weak homotopy equivalence,

$$(11) \quad B_\infty(W)^{\langle P, m \rangle} \sim \text{BDiff}(W)^{\langle P \rangle},$$

where $\text{BDiff}(W)^{\langle P \rangle}$ is the classifying space of the topological group $\text{Diff}(W)^{\langle P \rangle}$.

Using the well-known Borel construction we define,

$$(12) \quad E_\infty(W)^{\langle P, m \rangle} := \text{Emb}(W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{p+m})^{\langle P, m \rangle} \times_{\text{Diff}(W)^{\langle P \rangle}} W.$$

There is a fibre-bundle,

$$E_\infty(W)^{\langle P, m \rangle} \longrightarrow B_\infty(W)^{\langle P, m \rangle},$$

with fibre W and structure group $\text{Diff}(W)^{\langle P \rangle}$. This fibre-bundle (12) comes with a natural embedding

$$E_\infty(W)^{\langle P, m \rangle} \subset B_\infty(W)^{\langle P, m \rangle} \times [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{p+m}.$$

With this embedding the bundle is universal in the following sense: If $f : X \longrightarrow B_\infty(W)^{\langle P, m \rangle}$ is a smooth map from a smooth manifold X of dimension k , then the pullback

$$f^*(E_\infty(W)^{\langle P, m \rangle}) = \{(x, v) \in X \times ([0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{p+m}) \mid (f(x), v) \in E_\infty(W)^{\langle P, m \rangle}\}$$

is a smooth $(k + d + 1)$ -dimensional submanifold

$$E \subseteq X \times [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{p+m}$$

such that the projection onto X is a fibre-bundle with fibre W and structure group $\text{Diff}(W)^{\langle P \rangle}$. Furthermore, any such embedded fibre-bundle

$$E \subseteq X \times [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{p+m}$$

over X is induced by a unique smooth map $f : X \longrightarrow B_\infty(W)^{\langle P, m \rangle}$.

The submanifolds $\partial_0 W$, $\partial_1 W$, and $\partial_{0,1} W$ are invariant under the action of $\text{Diff}(W)^{\langle P \rangle}$ by definition. Therefore there are associated bundles,

$$\begin{aligned} E_\infty(\partial_0 W)^{\langle P \rangle} &:= \text{Emb}(W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{p+m})^{\langle P, m \rangle} \times_{\text{Diff}(W)^{\langle P \rangle}} \partial_0 W, \\ (13) \quad E_\infty(\partial_1 W)^{\langle P \rangle} &:= \text{Emb}(W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{p+m})^{\langle P, m \rangle} \times_{\text{Diff}(W)^{\langle P \rangle}} \partial_1 W, \\ E_\infty(\partial_{0,1} W)^{\langle P \rangle} &:= \text{Emb}(W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{p+m})^{\langle P, m \rangle} \times_{\text{Diff}(W)^{\langle P \rangle}} \partial_{0,1} W. \end{aligned}$$

Again, $\text{Diff}(W)^{\langle P \rangle}$ also acts on the manifolds $\beta_1 W$ and $\beta_{0,1} W$. It is easy to see that there are bundle-isomorphisms:

$$\begin{aligned} (14) \quad E_\infty(\partial_1 W)^{\langle P \rangle} &\cong (\text{Emb}(W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{p+m})^{\langle P, m \rangle} \times_{\text{Diff}(W)^{\langle P \rangle}} \beta_1 W) \times P, \\ E_\infty(\partial_{0,1} W)^{\langle P \rangle} &\cong (\text{Emb}(W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{p+m})^{\langle P, m \rangle} \times_{\text{Diff}(W)^{\langle P \rangle}} \beta_{0,1} W) \times P, \end{aligned}$$

that cover the identity map on the base space $B_\infty(W)^{\langle P \rangle}$. This leads us to:

Proposition 3.4. *Let X be a manifold without boundary. Suppose $W \longrightarrow E \longrightarrow X$ is a fibre bundle with P -manifold fibres diffeomorphic to W and structure group $\text{Diff}(W)^{\langle P \rangle}$. Then the total space E is itself a P -manifold.*

Proof. It is easy to see that the total-space of a fibre-bundle, with base space a manifold without boundary and fibre a manifold with corners, is itself a manifold with corners. The result follows by combining this fact with the factorization in (14) for the universal case. \square

4. THE COBORDISM CATEGORY

Let P be a closed manifold of dimension p . Fix an embedding

$$(15) \quad i_P : P \longrightarrow \mathbb{R}^{p+m}.$$

We will construct a category $\mathbf{Cob}_{d+1}^{P, m}$ with objects embedded d -dimensional P -manifolds with empty ∂_0 -component and morphisms embedded $(d+1)$ -dimensional P -manifolds.

Roughly, an object of $\mathbf{Cob}_{d+1}^{P, m}$ is a pair (M, a) where $a \in \mathbb{R}$ and $M \subseteq \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{p+m}$ is a compact d -dimensional P -submanifold with $\partial_0 M = \emptyset$. It is required that there exists $\epsilon > 0$ such that

$$M \cap ([0, \epsilon) \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{p+m}) = [0, \epsilon) \times \partial_1 M = [0, \epsilon) \times \beta_1 M \times P$$

where $P \subset \mathbb{R}^{p+m}$ is the submanifold specified in (15). A non-identity morphism of $\mathbf{Cob}_{d+1}^{P, m}$ from (M_a, a) to (M_b, b) is a triple $(W; a, b)$ with $(a, b) \in \mathbb{R}_+^2$ (here $\mathbb{R}_+^2 = \{(a, b) \in \mathbb{R}^2 \mid a < b\}$)

and

$$W \subseteq [a, b] \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{p+m}$$

a $(d+1)$ -dimensional compact P -submanifold with $\partial_0 W = M_a \sqcup M_b$. It is required that there exist $\epsilon_0, \epsilon_1 > 0$ such that

$$\begin{aligned} W \cap ([a, a + \epsilon_0] \sqcup (b - \epsilon_0, b]) \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{p+m} &= ([a, a + \epsilon_0] \sqcup (b - \epsilon_0, b]) \times \partial_0 W, \\ W \cap ([a, b] \times [0, \epsilon_1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{p+m}) &= [0, \epsilon_1] \times \partial_1 W = [0, \epsilon_1] \times \beta_1 W \times P \end{aligned}$$

where $P \subset \mathbb{R}^{p+m}$ is again the submanifold from 15. The morphisms $(W_1; a, b)$ and $(W_2; b, c)$ can be composed if

$$W_1 \cap (\{b\} \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{p+m}) = W_2 \cap (\{b\} \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{p+m}).$$

In this case their composition is given by $(W_1 \cup W_2; a, c)$. The collars from the definition ensure that this union is a manifold with a canonical smooth structure induced by the smooth structures from W_1 and W_2 . The identity morphisms are identified with the set of objects.

As sets, we have

$$\begin{aligned} \mathcal{O}b(\mathbf{Cob}_{d+1}^{P,m}) &\cong \bigsqcup_M (B_\infty(M)^{\langle P, m \rangle} \times \mathbb{R}), \\ (16) \quad \mathcal{M}or(\mathbf{Cob}_{d+1}^{P,m}) &\cong \mathcal{O}b(\mathbf{Cob}_{d+1}^{P,m}) \sqcup \left(\bigsqcup_W (\mathbb{R}_+^2 \times B_\infty(W)^{\langle P \rangle}) \right) \end{aligned}$$

where M varies over diffeomorphism classes of d -dimensional compact P -manifolds with $\partial_0 M = \emptyset$, W varies over diffeomorphism classes of $(d+1)$ -dimensional compact P -manifolds, and $\mathbb{R}_+^2 := \{(a, b) \in \mathbb{R}^2 \mid a < b\}$. Using this identification, $\mathbf{Cob}_{d+1}^{P,m}$ is made into a topological category. Each component having the structure of a smooth infinite dimensional manifold.

Our main theorem to prove is about the homotopy type of the classifying space $\mathbf{BCob}_{d+1}^{P,m}$. In order to determine the homotopy type of this space we will need to study a sheaf with representing space equivalent to $\mathbf{BCob}_{d+1}^{P,m}$. This will occupy our next section.

5. THE SHEAF $\mathbf{D}_{d+1}^{P,m}$

Let \mathcal{X} denote the category of smooth manifolds without boundary with morphisms given by smooth maps. By a sheaf (set valued) on \mathcal{X} we mean a contravariant functor \mathcal{F} from \mathcal{X} to **Sets** which satisfies the following condition. For any good open covering $\{U_i \mid i \in \Lambda\}$ of some $X \in \mathcal{O}b(\mathcal{X})$, and every collection $s_i \in \mathcal{F}(U_i)_i$ satisfying $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $i, j \in \Lambda$, there is a unique $s \in \mathcal{F}(X)$ such that $s|_{U_i} = s_i$ for all $i \in \Lambda$. This is the same definition used in [11].

Let P be a closed p -dimensional manifold. We fix an embedding,

$$(17) \quad i_P : P \longrightarrow \mathbb{R}^{p+m}.$$

We define a sheaf $\mathbf{D}_{d+1}^{P,m}$ on \mathcal{X} as follows:

Definition 5.1. Let $X \in \mathcal{Ob}(\mathcal{X})$. For $\epsilon > 0$, integers n and d such that $n > m$ and $d > p$, and $X \in \mathcal{Ob}(\mathcal{X})$ we define $\mathbf{D}_{d+1,n,\epsilon}^{P,m}(X)$ to be the set of triples $(W; \pi, f)$ where

$$W \subset X \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n-p-m} \times \mathbb{R}^{p+m},$$

is a $d+1$ -dimensional submanifold with boundary and $(\pi, f) : W \rightarrow X \times \mathbb{R}$ is the restriction to W of the projection onto $X \times \mathbb{R}$ (with X and \mathbb{R} viewed as the first and second factors in the above product), subject to the following conditions:

i. The boundary ∂W , which is given as

$$\partial W := W \cap (X \times \mathbb{R} \times \{0\} \times \mathbb{R}^{d-1+n-p-m} \times \mathbb{R}^{p+m}),$$

factors as a product $\partial W = \beta W \times P$. The subspace $\beta W \subset X \times \mathbb{R} \times \{0\} \times \mathbb{R}^{d-1+n-p-m}$ a submanifold and $P \subset \mathbb{R}^{p+m}$ the submanifold given by (17).

ii. The boundary ∂W is embedded with an ϵ -width closed collar neighborhood denoted by $N_\epsilon(\partial W)$, such that

$$N_\epsilon(\partial W) = W \cap (X \times \mathbb{R} \times [0, \epsilon] \times \mathbb{R}^{d-1+n-p-m} \times \mathbb{R}^{p+m}) = \partial W \times [0, \epsilon].$$

iii. The map π and restriction $\pi|_{\partial W}$ are both submersions with $d+1$ and d -dimensional fibres respectively. Furthermore the restriction of π to the collar $N_\epsilon(\partial W)$ has the factorization,

$$N_\epsilon(\partial W) = \partial W \times [0, \epsilon] \xrightarrow{pr} \partial W = \beta W \times P \xrightarrow{pr} \beta W \xrightarrow{\pi_{\beta W}} X$$

where pr denotes projection and where the map $\pi_{\beta W}$ is a submersion as well.

iv. The map (π, f) is a proper map. Furthermore the restriction of f to the collar $N_\epsilon(\partial W)$ has the factorization,

$$N_\epsilon(\partial W) = \partial W \times [0, \epsilon] \xrightarrow{pr} \partial W = \beta W \times P \xrightarrow{pr} \beta W \xrightarrow{f_{\beta W}} \mathbb{R}$$

where $(\pi_{\beta W}, f_{\beta W})$ is a smooth-proper map as well.

It can be verified that the functor $\mathbf{D}_{d+1,n,\epsilon}^{P,m}$ satisfies the sheaf-condition on all elements $X \in \mathcal{X}$.

Remark 5.1. Notice that any for an element $(W; \pi, f) \in \mathbf{D}_{d+1,n,\epsilon}^{P,m}(X)$, W is a P -submanifold; non-compact in general. There is no ∂_0 component and so we omit subscripts on the symbol ∂ since there is no issue of ambiguity.

For all $n > 0$ and $\epsilon_2 \geq \epsilon_1$ there is an obvious map of sheaves

$$(18) \quad \mathbf{D}_{d+1,n,\epsilon_1}^{P,m} \longrightarrow \mathbf{D}_{d+1,n+1,\epsilon_2}^{P,m}.$$

Using this we define,

$$(19) \quad \mathbf{D}_{d+1}^{P,m} := \operatorname{colim}_{\substack{\epsilon \rightarrow 0 \\ n \rightarrow \infty}}^* \mathbf{D}_{d+1,n,\epsilon}^{P,m}.$$

By $\text{colim}_{\substack{\epsilon \rightarrow 0 \\ n \rightarrow \infty}}^*$ we mean the sheaffication of $\text{colim}_{\substack{\epsilon \rightarrow 0 \\ n \rightarrow \infty}} \mathbf{D}_{d+1,n}^{P,m}$ where this direct limit with no superscript of $*$ is the direct limit taken in the category of pre-sheaves. Notice that prior to sheaffication, the sheaf gluing condition may fail for the pre-sheaf direct limit $\text{colim}_{\substack{\epsilon \rightarrow 0 \\ n \rightarrow \infty}} \mathbf{D}_{d+1,n}^{P,m}$ when applied to infinite covers. Indeed, let X be a non-compact space and let $U_i, i \in \mathbb{Z}_+$ be an infinite cover with no finite sub-cover. It is quite possible to find compatible elements $(W_i; \pi_i, f_i) \in \mathbf{D}_{d+1,n_i,\epsilon_i}^{P,m}$ with $\lim_{i \rightarrow \infty} n_i = \infty$ such that the resulting manifold obtained by gluing together the W_i does not embed into

$$X \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n}$$

for any n . There also may not exist a constant $\epsilon > 0$ such that W is embedded with collar of length ϵ . Such a manifold will embed into $X \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty}$ however and will have a collar whose length is positive constant function of W as opposed to a constant. In the case that X is compact, no such issue with gluing arises since only finite covers need be considered. Using this observation we get an explicit description of $\mathbf{D}_{d+1}^{P,m}(X)$ for X a compact manifold which will enable us to manipulate elements of $\mathbf{D}_{d+1}^{P,m}(X)$.

Lemma 5.1. *For X a compact manifold without boundary, the set $\mathbf{D}_{d+1}^{P,m}(X)$ as defined in (19) is equal to the set*

$$\text{colim}_{\substack{\epsilon \rightarrow 0 \\ n \rightarrow \infty}} \mathbf{D}_{d+1,n,\epsilon}^{P,m}(X)$$

where the direct limit is taken in the category of sets.

We will at times want to work with $\mathbf{D}_{d+1,n,\epsilon}^{P,m}$ prior to taking the direct limit but will want to eliminate dependence on ϵ . To this end we denote,

$$(20) \quad \mathbf{D}_{d+1,n}^{P,m} := \text{colim}_{\epsilon \rightarrow 0}^* \mathbf{D}_{d+1,n,\epsilon}^{P,m}.$$

We will ultimately be interested in the concordance classes of $\mathbf{D}_{d+1}^{P,m}$. We now recall some important definitions regarding the concordance theory of sheaves.

Definition 5.2. Let \mathcal{F} be a sheaf on \mathcal{X} . Two elements s_0 and s_1 of $\mathcal{F}(X)$ are said to be concordant if there exists $s \in \mathcal{F}(X \times \mathbb{R})$ that agrees with $pr^*(s_0)$ in an open neighborhood of $X \times (-\infty, 0]$ and agrees with $pr^*(s_1)$ in an open neighborhood of $X \times [1, \infty)$.

We denote the set of concordance classes of $\mathcal{F}(X)$ by $\mathcal{F}[X]$. The correspondence $X \mapsto \mathcal{F}[X]$ is clearly functorial in X . Now, recall the definition of the *representing space* of \mathcal{F} , denoted by $|\mathcal{F}|$. The space $|\mathcal{F}|$ is defined to be the geometric realization of the simplicial set given by $k \mapsto \mathcal{F}(\Delta_e^k)$ where

$$\Delta_e^k := \{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} \mid \sum x_i = 1\}$$

is the standard extended k -simplex. In [11] it is shown that the functors given by

$$X \mapsto \mathcal{F}[X] \quad \text{and} \quad X \mapsto [X, |\mathcal{F}|]$$

(where $[X, |\mathcal{F}|]$ denotes the set of homotopy classes of maps from X to $|\mathcal{F}|$) are naturally isomorphic.

Definition 5.3. A map of sheaves $\mathcal{F} \rightarrow \mathcal{G}$ is said to be a *weak equivalence* if it induces a homotopy equivalence

$$|\mathcal{F}| \sim |\mathcal{G}|$$

of representing spaces.

For details on the concordance theory of sheaves and relevant constructions we refer to [11].

We now state two useful lemmas relevant to the sheaf $\mathbf{D}_{d+1}^{P,m}$.

Lemma 5.2. *Every concordance class in $\mathbf{D}_{d+1,n}^{P,m}$ has a representative $(W; \pi, f)$ with*

$$f : W \rightarrow \mathbb{R}$$

a bundle projection, so that

$$W \cong f^{-1}(0) \times \mathbb{R}.$$

Proof. The proof is identical to that of [11, 2.5.2] □

Lemma 5.3. *The map of representing spaces*

$$|\mathbf{D}_{d+1,n}^{P,m}| \rightarrow |\mathbf{D}_{d+1,n+1}^{P,m}|$$

induced by (18) yields an isomorphism $\pi_k(|\mathbf{D}_{d+1,n}^{P,m}|) \cong \pi_k(|\mathbf{D}_{d+1,n+1}^{P,m}|)$ whenever

$$2k + 2d + 5 + p + m < n.$$

Proof. We give the proof in Appendix C □

From this result and Lemma 5.1 we get

Corollary 5.4. *There is a homotopy equivalence*

$$|\mathbf{D}_{d+1}^{P,m}| \sim \operatorname{colim}_{\substack{\epsilon \rightarrow 0 \\ n \rightarrow \infty}} |\mathbf{D}_{d+1,n,\epsilon}^{P,m}|.$$

6. A COFIBRE OF THOM-SPECTRA

In what follows we will define a spectrum $\mathbf{MT}(d+1)_{\langle P, m, \partial \rangle}$ as the cofibre of a map between $\Sigma^{-1}\mathbf{MT}(d)$ and $\mathbf{MT}(d+1)$, where $\mathbf{MT}(d)$ is the spectrum defined in [6]. We use the same notation for Grassmanian manifolds and their canonical bundles as in [6].

We consider the same embedding $i_P : P \hookrightarrow \mathbb{R}^{p+m}$ as in (17). The normal bundle, N_P of P has Gauss map,

$$\begin{array}{ccc} N_P & \xrightarrow{\hat{\gamma}} & U_{p,m}^\perp \\ \downarrow & & \downarrow \\ P & \xrightarrow{\gamma} & G(p, m) \end{array}$$

which induces a map of the Thom spaces, $\mathrm{Th}(\hat{\gamma}) : \mathrm{Th}(N_p) \longrightarrow \mathrm{Th}(U_{p,m}^\perp)$. Now, we fix a tubular neighborhood embedding,

$$(21) \quad e_P : N_P \hookrightarrow \mathbb{R}^{p+m}.$$

This tubular neighborhood together with $\mathrm{Th}(\hat{\gamma})$ yields the Thom-Pontryagin map

$$(22) \quad S^{p+m} \xrightarrow{c_P} \mathrm{Th}(U_{p,m}^\perp).$$

We now consider the “multiplication-map”

$$\mu : G(d-p, n-m) \times G(p, m) \longrightarrow G(d, n)$$

given by

$$(V, W) \longmapsto V \oplus W.$$

The pullback bundle $\mu^*(U_{d,n}^\perp)$ over $G(d-p, n-m) \times G(p, m)$, can be identified with the product bundle $U_{d-p, n-m}^\perp \times U_{p,m}^\perp$. We then get the bundle map

$$\begin{array}{ccc} U_{d-p, n-m}^\perp \times U_{p,m}^\perp & \xrightarrow{\hat{\mu}} & U_{d,n}^\perp \\ \downarrow & & \downarrow \\ G(d-p, n-m) \times G(p, m) & \xrightarrow{\mu} & G(d, n) \end{array}$$

which induces the map of Thom spaces

$$\mathrm{Th}(U_{d-p, n-m}^\perp) \wedge \mathrm{Th}(U_{p,m}^\perp) \xrightarrow{\mathrm{Th}(\hat{\mu})} \mathrm{Th}(U_{d,n}^\perp).$$

Putting this together with c_P from (22) we define:

$$(23) \quad \tau_{P,m}^n := \mathrm{Th}(\hat{\mu}) \circ (c_P \wedge \mathrm{Id}) : \mathrm{Th}(U_{d-p, n-m}^\perp) \wedge S^{p+m} \longrightarrow \mathrm{Th}(U_{d,n}^\perp).$$

It is well known that the space $\mathrm{Th}(U_{d,n}^\perp)$ is the $(d+n)$ th space of the spectrum $\mathbf{MT}(d)$. The structure maps in this spectrum $\mathbf{MT}(d)$ come from maps induced by the bundle maps

$$\begin{array}{ccc} U_{d,n}^\perp \oplus \epsilon^1 & \xrightarrow{\hat{i}_n} & U_{d,n+1}^\perp \\ \downarrow & & \downarrow \\ G(d, n) & \xrightarrow{i_n} & G(d, n+1) \end{array}$$

where the map i_n is the standard embedding. The map from (23), yields a map of spectra

$$(24) \quad \tau_{P,m} : \mathbf{MT}(d-p) \longrightarrow \mathbf{MT}(d)$$

where we are taking into account that the spectrum with $(d+n)$ th space $\mathrm{Th}(U_{d-p, n-m}^\perp) \wedge S^{p+m}$ is homotopy equivalent to $\mathbf{MT}(d)$. We define,

$$(25) \quad \mathbf{MT}(d)_{\langle P,m \rangle} := \mathrm{Cofibre}(\tau_{P,m} : \mathbf{MT}(d-p) \longrightarrow \mathbf{MT}(d)).$$

Let

$$j_d^n : G(d, n) \longrightarrow G(d+1, n)$$

be the map given as follows: Let $e_{d-1+(n+1)-p-m}$ be the $(d-1+(n+1)-p-m)$ th standard basis vector for the factor of $\mathbb{R}^{d-1+(n+1)-p-m}$ in the space $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d-1+(n+1)-p-m} \times \mathbb{R}^{p+m}$. Let

$$V \subset \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d-1+n-p-m} \times \mathbb{R}^{p+m}$$

be a d -dimensional subspace. We define,

$$(26) \quad j_d^n(V) := V \oplus \langle e_{d-1+(n+1)-p-m} \rangle \subset \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d-1+(n+1)-p-m} \times \mathbb{R}^{p+m}.$$

The map j_d^n induces a bundle map $U_{d,n}^\perp \longrightarrow U_{d+1,n}^\perp$. The induced map of the Thom spaces, which we denote by \tilde{j}_d^n , yields a map of spectra

$$\Sigma^{-1}\mathbf{MT}(d) \xrightarrow{\tilde{j}_d} \mathbf{MT}(d+1).$$

We define,

$$(27) \quad \mathbf{MT}(d+1)_{\langle \partial \rangle} := \text{Cofibre}(\tilde{j}_d : \Sigma^{-1}\mathbf{MT}(d) \longrightarrow \mathbf{MT}(d+1)).$$

We now consider the composition of $\Sigma^{-1}\tau_P$ with \tilde{j}_d . We define,

$$(28) \quad \mathbf{MT}(d+1)_{\langle P, m, \partial \rangle} := \text{Cofibre}(\tilde{j}_d \circ (\Sigma^{-1}\tau_P) : \Sigma^{-1}\mathbf{MT}(d-p) \longrightarrow \mathbf{MT}(d+1)).$$

Remark 6.1. From the three spectra just defined the one that shows up in the statement of Theorem 1.1 is $\mathbf{MT}(d+1)_{\langle P, m, \partial \rangle}$. However the other two have interesting interpretations as well. In [6] it is shown that there is a homotopy equivalence, $\mathbf{MT}(d+1)_{\langle \partial \rangle} \sim \Sigma^\infty \mathbf{BO}(d+1)$. In the case where P equals a single point denoted by \star , the map τ_P^m is the identity map and Theorem 1.1 reduces to the weak homotopy equivalence

$$\mathbf{BCob}_{d+1}^\star \sim \Omega^{\infty-1}\mathbf{MT}(d+1)_{\langle \partial \rangle} \sim \Omega^{\infty-1}\Sigma^\infty \mathbf{BO}(d+1).$$

From the composition

$$\Sigma^{-1}\mathbf{MT}(d-p) \xrightarrow{\tau_P} \Sigma^{-1}\mathbf{MT}(d) \xrightarrow{\tilde{j}_{d-p}} \mathbf{MT}(d+1),$$

we see that these three spectra and their infinite loopspaces fit together via the cofibre sequence,

$$\mathbf{MT}(d+1)_{\langle P, m, \partial \rangle} \longrightarrow \mathbf{MT}(d+1)_{\langle \partial \rangle} \longrightarrow \mathbf{MT}(d)_{\langle P, m \rangle}$$

and homotopy fibre sequence,

$$\Omega^{\infty-1}\mathbf{MT}(d+1)_{\langle P, m, \partial \rangle} \longrightarrow \Omega^{\infty-1}\mathbf{MT}(d+1)_{\langle \partial \rangle} \longrightarrow \Omega^{\infty-1}\mathbf{MT}(d)_{\langle P, m \rangle}.$$

There is a direct system of spectra

$$(29) \quad \dots \longrightarrow \Sigma^{(d-1)}\mathbf{MT}(d-1) \longrightarrow \Sigma^d\mathbf{MT}(d) \longrightarrow \Sigma^{(d+1)}\mathbf{MT}(d+1) \longrightarrow \dots$$

where the d -th map is $\Sigma^d \tilde{j}_d$. The direct limit is a spectrum which we denote by \mathbf{MTO} .

Lemma 6.1. *There is a homotopy equivalence*

$$\mathbf{MTO} \sim \mathbf{MO}$$

Proof. There is a homeomorphism $G(d, n) \rightarrow G(n, d)$ given by $V \mapsto V^\perp$. This map is covered by a bundle isomorphism $U_{d,n}^\perp \rightarrow U_{n,d}$ and thus gives us maps

$$\mathbf{Th}(U_{d,n}^\perp) \xrightarrow[\cong]{\perp} \mathbf{Th}(U_{n,d}) \xrightarrow{i} \mathbf{Th}(U_{n,\infty}).$$

This map is equivalent to

$$\mathbf{Th}(U_{d,n}^\perp) \xrightarrow{\tilde{j}_d^n} \mathbf{Th}(U_{d+1,n}^\perp) \xrightarrow[\cong]{\perp} \mathbf{Th}(U_{n,d+1}) \xrightarrow{i} \mathbf{Th}(U_{n,\infty}).$$

The space $\mathbf{Th}(U_{n,\infty})$ is the n -th space in the spectrum \mathbf{MO} . Now, $\mathbf{Th}(U_{n,\infty})$ can be built out of $\mathbf{Th}(U_{n,d})$ by attaching cells of dimension greater than dimension $n + d$. This implies that the resulting map of spectra, $\Sigma^d \mathbf{MT}(d) \rightarrow \mathbf{MO}$, induces an isomorphism on π_k for $k < d$ and a surjection for $k = d$. This proves that $\mathbf{MTO} \sim \mathbf{MO}$. \square

It is known that $\pi_{d-1} \mathbf{MO} = \Omega_{d-1}$ where Ω_{d-1} is the cobordism group of unoriented $(d-1)$ -dimensional manifolds. The above proof implies that $\pi_{-1} \mathbf{MT}(d) = \Omega_{d-1}$.

Now, for each d , the diagram

$$\begin{array}{ccc} \Sigma^{(d+p)} \mathbf{MT}(d) & \xrightarrow{\Sigma^{(d+p)} \tilde{j}_d} & \Sigma^{(d+1+p)} \mathbf{MT}(d+1) \\ \downarrow \Sigma^{(d+1+p)} [\tilde{j}_d \circ (\Sigma^{-1} \tau_{P,m})] & & \downarrow \Sigma^{(d+2+p)} [\tilde{j}_{d+1} \circ (\Sigma^{-1} \tau_{P,m})] \\ \Sigma^{(d+1+p)} \mathbf{MT}(d+1+p) & \xrightarrow{\Sigma^{(d+1+p)} \tilde{j}_{d+1+p}} & \Sigma^{(d+2+p)} \mathbf{MT}(d+2+p) \end{array}$$

commutes up to homotopy. This induces a map of spectra

$$(30) \quad \Sigma^{d+1+p} \mathbf{MT}(d+1+p)_{\langle P, m, \partial \rangle} \longrightarrow \Sigma^{d+2+p} \mathbf{MT}(d+2+p)_{\langle P, m, \partial \rangle}$$

for each d . These maps form a direct system similar to (29). We denote its direct limit by $\mathbf{MTO}_{\langle P, m, \partial \rangle}$.

Lemma 6.2. *There is a homotopy equivalence*

$$\mathbf{MTO}_{\langle P, m, \partial \rangle} \sim \mathbf{MO}_P$$

where \mathbf{MO}_P is the classifying spectrum for the cobordism theory Ω_*^P for manifolds with type P -singularity mentioned in the introduction.

Proof. The spectrum \mathbf{MO}_P is given as the cofibre of a map $\times P : \Sigma^p \mathbf{MO} \longrightarrow \mathbf{MO}$. This map is constructed as follows. The map

$$\mu : G(n, d) \times G(m, p) \longrightarrow G(n + m, d + p)$$

given by $(V, W) \mapsto V \oplus W$, induces a map

$$\mu' : G(n, \infty) \times G(m, p) \longrightarrow G(n + m, \infty)$$

in the limit as $d \rightarrow \infty$. The map μ' is covered by a bundle map $U_{n, \infty} \times U_{m, p} \longrightarrow U_{n+m, \infty}$ which induces a map

$$\mathbf{Th}(U_{n, \infty}) \wedge \mathbf{Th}(U_{m, p}) \longrightarrow \mathbf{Th}(U_{n+m, \infty}).$$

The normal bundle N_P for $P \subset \mathbb{R}^{p+m}$ can be given a Gauss map

$$\begin{array}{ccc} N_P & \longrightarrow & U_{m, p} \\ \downarrow & & \downarrow \\ P & \longrightarrow & G(m, p). \end{array}$$

We emphasize that this map is different that the one given earlier where the target space was $G(p, m)$ with bundle $U_{p, m}^\perp$. Using the Pontryagin-Thom map, $S^{p+m} \longrightarrow \mathbf{Th}(U_{m, p})$ we get a map

$$\mathbf{Th}(U_{n, \infty}) \wedge S^{p+m} \longrightarrow \mathbf{Th}(U_{n, \infty}) \wedge \mathbf{Th}(U_{m, p}) \longrightarrow \mathbf{Th}(U_{n+m, \infty}).$$

Taking into account that the spectrum with $(n + m)$ -th space equal to $\mathbf{Th}(U_{n, \infty}) \wedge S^m$ is homotopy equivalent to \mathbf{MO} , this map above induces a map of spectra

$$\Sigma^p \mathbf{MO} \longrightarrow \mathbf{MO}$$

which defines $\times P$. Upon inspection, it can be seen that the following diagram commutes

$$(31) \quad \begin{array}{ccc} \Sigma^{d+p} \mathbf{MT}(d) & \longrightarrow & \Sigma^p \mathbf{MO} \\ \downarrow \tilde{j}_d \circ \tau_{P, m} & & \downarrow \times P \\ \Sigma^{d+p+1} \mathbf{MT}(d + p + 1) & \longrightarrow & \mathbf{MO} \end{array}$$

where the horizontal maps are induced by

$$\mathbf{Th}(U_{d, n}^\perp) \xrightarrow[\cong]{\perp} \mathbf{Th}(U_{n, d}) \xrightarrow{i} \mathbf{Th}(U_{n, \infty}).$$

According to Lemma 6.1, the Thom space $\mathbf{Th}(U_{n, \infty})$ can be built out of $\mathbf{Th}(U_{n, d})$ by attaching cells of dimension greater than dimension $n + d$. This implies that the lower and upper horizontal maps above in (31) induce isomorphisms on π_k for $k < d + p + 1$ and surjections on π_k for $k = d + p + 1$. By applying the *Five Lemma* to the long exact sequence on homotopy groups associated to the cofibres of the vertical maps, we see that the induced map

$$\Sigma^{d+p+1} \mathbf{MT}(d + p + 1)_{\langle P, m, \partial \rangle} \longrightarrow \mathbf{MO}_P$$

induces an isomorphism on π_k for $k < d + p + 1$ and a surjection on π_k for $k = d + p + 1$. Taking the direct limit as $d \rightarrow \infty$, we see that $\mathbf{MTO}_{\langle P, m, \partial \rangle} \sim \mathbf{MO}_P$. \square

Corollary 6.3. *There is an isomorphism $\pi_{-1}\mathbf{MT}(d+1)_{\langle P,m,\partial \rangle} = \Omega_d^P$.*

7. THE MAIN THEOREM

Theorem 7.1. [Main Theorem] *There is a homotopy equivalence*

$$|\mathbf{D}_{d+1}^{P,m}| \sim \Omega^{\infty-1}\mathbf{MT}(d+1)_{\langle P,m,\partial \rangle}.$$

7.1. Isomorphism of Concordance Class Functors. In our first step to proving Theorem 7.1 we will prove:

Lemma 7.2. *For any compact manifold X there is an isomorphism of sets*

$$(32) \quad \mathbf{D}_{d+1}^{P,m}[X] \longrightarrow [X, \Omega^{\infty-1}\mathbf{MT}(d+1)_{\langle P,m,\partial \rangle}].$$

This isomorphism is natural with respect to maps between compact manifolds.

To prove this we need to make another definition:

Definition 7.1. We define $\Omega_{\langle P,\partial \rangle}^{d+n} \mathbf{Th}(U_{d+1,n}^\perp)_{\langle P,m,\partial \rangle}$ to be the space of pairs (\tilde{f}, f) of based maps,

$$\begin{aligned} \tilde{f} : D^{d+n} &\longrightarrow \mathbf{Th}(U_{d+1,n}^\perp), \\ f : S^{d+n-1-m-p} \wedge S^{p+m} &\longrightarrow \mathbf{Th}(U_{d-p,n-m}^\perp) \wedge S^{p+m} \end{aligned}$$

which satisfy the following conditions:

- i. The following diagram commutes

$$\begin{array}{ccc} D^{d+n} & \xrightarrow{\tilde{f}} & \mathbf{Th}(U_{d+1,n}^\perp) \\ \uparrow & & \uparrow \tilde{j}_d^n \circ \tau_{P,m}^n \\ S^{d+n-1-p-m} \wedge S^{p+m} & \xrightarrow{f} & \mathbf{Th}(U_{d-p,n-m}^\perp) \wedge S^{p+m} \end{array}$$

where the left vertical map is given by identifying $S^{d-1+n-p-m} \wedge S^{p+m}$ with S^{d-1+n} and then including into D^{d-1+n} as boundary.

- ii. The map f is given the factorization, $f = f_0 \wedge Id_{S^{p+m}}$ where f_0 is a map from $S^{d-1+n-p-m}$ to $\mathbf{Th}(U_{d-p,n-m}^\perp)$.

The space $\Omega_{\langle P,\partial \rangle}^{d+n} \mathbf{Th}(U_{d+1,n}^\perp)_{\langle P,m,\partial \rangle}$ is then topologized as a subspace of

$$\mathcal{C}^0(D^{d+n}, \mathbf{Th}(U_{d+1,n}^\perp)) \times \mathcal{C}^0(S^{d+n-1}, \mathbf{Th}(U_{d-p,n-m}^\perp) \wedge S^{p+m}).$$

For each n there is an inclusion

$$\Omega_{\langle P,\partial \rangle}^{d+n} \mathbf{Th}(U_{d+1,n}^\perp)_{\langle P,m,\partial \rangle} \longrightarrow \Omega_{\langle P,\partial \rangle}^{d+n+1} \mathbf{Th}(U_{d+1,n+1}^\perp)_{\langle P,m,\partial \rangle}.$$

We define,

$$(33) \quad \Omega_{\langle P, \partial \rangle}^{\infty-1} \mathbf{MT}(d+1)_{\langle P, m, \partial \rangle} := \operatorname{colim}_{n \rightarrow \infty} \Omega_{\langle P, \partial \rangle}^{d+n} \mathbf{Th}(U_{d+1, n}^\perp)_{\langle P, m, \partial \rangle}.$$

Notice that each element of $\Omega_{\langle P, \partial \rangle}^{d+n} \mathbf{Th}(U_{d+1, n}^\perp)_{\langle P, m, \partial \rangle}$ induces a unique map

$$S^{d+n} \rightarrow \operatorname{Cofibre}(\tilde{j}_d^n \circ \tau_{P, m}^n).$$

Thus, each for each n there is a map

$$\sigma_n : \Omega_{\langle P, \partial \rangle}^{d+n} \mathbf{Th}(U_{d+1, n}^\perp)_{\langle P, m, \partial \rangle} \longrightarrow \Omega^{d+n} \operatorname{Cofibre}(\tilde{j}_d^n \circ \tau_{P, m}^n)$$

which induces a map,

$$(34) \quad \sigma : \Omega_{\langle P, \partial \rangle}^{\infty-1} \mathbf{MT}(d+1)_{\langle P, m, \partial \rangle} \longrightarrow \Omega^{\infty-1} \mathbf{MT}(d+1)_{\langle P, m, \partial \rangle}$$

in the limit. It will later be seen that σ is a homotopy equivalence.

Lemma 7.3. *For $X \in \mathcal{Ob}(\mathcal{X})$ there is a natural map*

$$T_m^n : \mathbf{D}_{d+1, n}^{P, m}[X] \longrightarrow [X, \Omega_{\langle P, \partial \rangle}^{d+n} \mathbf{Th}(U_{d+1, n}^\perp)_{\langle P, m, \partial \rangle}].$$

For compact X , the induced map

$$T_m : \mathbf{D}_{d+1}^{P, m}[X] \longrightarrow [X, \Omega_{\langle P, \partial \rangle}^{\infty-1} \mathbf{MT}(d+1)_{\langle P, m, \partial \rangle}]$$

is an isomorphism of sets.

Proof. Let $X \in \mathcal{Ob}(\mathcal{X})$. We construct a map

$$\mathbf{D}_{d+1, n}^{P, m}[X] \xrightarrow{T_m} [X, \Omega_{\langle P, \partial \rangle}^{d+n} \mathbf{Th}(U_{d+1, n}^\perp)_{\langle P, m, \partial \rangle}]$$

as follows. Let $(W; \pi, f)$ with

$$W \subset X \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n}$$

represent an element of $\mathbf{D}_{d+1, n}^{P, m}[X]$. The normal bundle for W which we denote by N_W , has Gauss map

$$(35) \quad \begin{array}{ccc} N_W & \xrightarrow{\gamma_{\hat{W}}} & U_{d+1, n}^\perp \\ \downarrow & & \downarrow \\ W & \xrightarrow{\gamma_W} & G(d+1, n). \end{array}$$

Since

$$\partial W = W \cap (X \times \mathbb{R} \times \{0\} \times \mathbb{R}^{d-1+n}) = \beta W \times P,$$

with

$$\beta W \subset X \times \mathbb{R} \times \{0\} \times \mathbb{R}^{d-1+n-p-m} \quad \text{and} \quad P \subset \mathbb{R}^{p+m},$$

the restriction $N_W|_{\partial W}$, factors as a product of normal bundles $N_{\beta W} \times N_P$ where $N_{\beta W}$ and N_P are the normal bundles for $\beta W \subset (X \times \mathbb{R} \times \{0\} \times \mathbb{R}^{d-1+n-p-m})$ and $P \subset \mathbb{R}^{p+m}$ respectively.

This factorization of normal bundles implies that the Gauss map $(\hat{\gamma}|_{\partial W}, \gamma_{\partial W})$ for the restriction $N_W|_{\partial W}$, has factorization

$$(\hat{\gamma}_W|_{\partial W}, \gamma_{\partial W}) = (\hat{\gamma}_{\beta W} \times \hat{\gamma}_P, \gamma_{\beta W} \times \gamma_P)$$

where $(\hat{\gamma}_{\beta W}, \gamma_{\beta W})$ and $(\hat{\gamma}_P, \gamma_P)$ are Gauss maps for $N_{\beta W}$ and N_P respectively.

By Sard's Theorem we may assume that both f and its restrictions $f|_{\partial W}$, $f|_{\beta W \times \{0\}}$ are transverse to 0. By transversality, the pre-images

$$M := f^{-1}(0), \quad \partial M := (f|_{\partial W})^{-1}(0), \quad \beta M := (f|_{\beta W \times \{0\}})^{-1}(0)$$

are manifolds. It also follows from the definition of f and how W was embedded that $\partial M = \beta M \times P$. The bundles $N_W, N_{\partial W}$, and $N_{\beta W}$ restrict to $N_M, N_{\partial M}$, and $N_{\beta M}$ where $N_M, N_{\partial M}$, and $N_{\beta M}$ are the normal bundles to $M, \partial M$, and βM as sub-manifolds of

$X \times \{0\} \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n}$, $X \times \{0\} \times \{0\} \times \mathbb{R}^{d-1+n}$, and $X \times \{0\} \times \{0\} \times \mathbb{R}^{d-1-p+n-m}$ respectively. These bundles yield Gauss maps,

$$(36) \quad \begin{array}{ccc} N_M & \xrightarrow{\gamma_M} & U_{d+1,n}^\perp \\ \downarrow & & \downarrow \\ M & \xrightarrow{\gamma_M} & G(d+1, n) \end{array}$$

and

$$(37) \quad \begin{array}{ccccccc} N_{\beta M} \times N_P & \xrightarrow{\gamma_{\beta M} \times \gamma_P} & U_{d-p,n-m}^\perp \times U_{p,m}^\perp & \xrightarrow{\hat{\mu}} & U_{d,n}^\perp \\ \downarrow & & \downarrow & & \downarrow \\ \beta M \times P & \xrightarrow{\gamma_{\beta M} \times \gamma_P} & G(d-p, n-m) \times G(p, m) & \xrightarrow{\mu} & G(d, n) \end{array}$$

which induce maps on Thom spaces,

$$\mathrm{Th}(N_M) \xrightarrow{\mathrm{Th}(\gamma_M)} \mathrm{Th}(U_{d+1,n}^\perp)$$

and

$$\mathrm{Th}(N_{\beta M}) \wedge \mathrm{Th}(N_P) \xrightarrow{\mathrm{Th}(\gamma_{\beta M}) \wedge \mathrm{Th}(\gamma_P)} \mathrm{Th}(U_{d-p,n-m}^\perp) \wedge \mathrm{Th}(U_{p,m}^\perp).$$

There are tubular neighborhood embeddings the normal bundles $N_M, N_{\partial M}$, and $N_{\beta M}$ into $X \times \{0\} \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n}$, $X \times \{0\} \times \{0\} \times \mathbb{R}^{d-1+n}$, and $X \times \{0\} \times \{0\} \times \mathbb{R}^{d-1-p+n-m}$ respectively which yield collapse maps,

$$\begin{aligned} \tilde{c}_M &: X \wedge D^{d+n} \rightarrow \mathrm{Th}(N_M) \\ \tilde{c}_{\partial M} &: X \wedge S^{d+n-1} \rightarrow \mathrm{Th}(N_{\partial M}), \\ \tilde{c}_{\beta M} &: X \wedge S^{d+n-1-p-m} \rightarrow \mathrm{Th}(N_{\beta M}). \end{aligned}$$

Composing with $\text{Th}(\hat{\gamma}_M)$ and $\text{Th}(\hat{\gamma}_{\beta M}) \wedge \text{Th}(\hat{\gamma}_P)$ we obtain a commutative diagram,

$$\begin{array}{ccc} X \wedge D^{d+n} & \xrightarrow{\quad} & \text{Th}(U_{d+1,n}^\perp) \\ \uparrow & & \uparrow \\ X \wedge S^{d+n-1-p-m} \wedge S^{p+m} & \xrightarrow{\quad} & \text{Th}(U_{d-p,n-p}^\perp) \wedge S^{p+m} \end{array}$$

where the top-horizontal map is given by $\text{Th}(\hat{\gamma}_{N_W}) \circ \tilde{c}_M$ and the bottom-horizontal map is given by $(\text{Th}(\hat{\gamma}_{N_{\beta M}}) \circ \tilde{c}_{\beta M}) \wedge Id_{S^{p+m}}$. The right-vertical map is given by

$$\tilde{j}_d^n \circ \text{Th}(\hat{\mu}) \circ [Id_{\text{Th}(U_{d-p,n-p}^\perp)} \wedge (\tilde{c} \circ \text{Th}(\hat{\gamma}_P))]$$

which is by definition equal to $\tilde{j}_d^n \circ \tau_{P,m}^n$. Using adjunction this commutative diagram yields a map

$$f : X \longrightarrow \Omega_{\langle P, \partial \rangle}^{d+n} \text{Th}(U_{d+1,n}^\perp)_{\langle P, m, \partial \rangle}.$$

By standard Pontryagin-Thom theory, it is easy to see that choosing a different representative of the concordance class of W would yield a map homotopic to the one which we just produced, just run the same process on a concordance. We then define $T_m^n([W]) := [f]$. It is easy to check that this definition is natural in the variable X .

We now now make the assumption that that X is compact. We construct an inverse to

$$T_m : \mathbf{D}_{d+1}^{P,m}[X] \longrightarrow [X, \Omega_{\langle P, \partial \rangle}^{\infty-1} \mathbf{MT}(d+1)_{\langle P, m, \partial \rangle}]$$

which we will denote by H_m . Let

$$\begin{array}{ccc} X \wedge D^{d+n} & \xrightarrow{\quad \tilde{f} \quad} & \text{Th}(U_{d+1,n}^\perp) \\ \uparrow & & \uparrow \tilde{j}_d^n \circ \tau_{P,m}^n \\ X \wedge S^{d+n-1-p-m} \wedge S^{p+m} & \xrightarrow{\quad f \quad} & \text{Th}(U_{d-1-p,n-m}^\perp) \wedge S^{p+m} \end{array}$$

represent an element of $[X, \Omega_{\langle P, \partial \rangle}^{d+n} \text{Th}(U_{d+1,n}^\perp)_{\langle P, m, \partial \rangle}]$.

By applying an appropriate homotopy, we may assume the following about (\tilde{f}, f) :

- i. The maps \tilde{f} and $\tau_{P,m}^n \circ f$ are both smooth and transverse to $G(d+1, n)$ and $G(d, n)$ as submanifolds of $U_{d+1,n}^\perp$ and $U_{d,n}^\perp$ respectively (notice that under the map j_d^n , $G(d, n)$ embeds as a submanifold of $G(d+1, n)$).
- ii. By transversality in (i.) we have a pair of submanifolds

$$(M, \partial M) \subseteq (X \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n}, X \times \mathbb{R}^{d-1+n})$$

where $M = \tilde{f}^{-1}(G(d+1, n))$ and $\partial M = (\tau_{P,m}^n \circ f)^{-1}(G(d, n)) = \partial \tilde{f}^{-1}(G(d+1, n))$. The compactness assumption on X implies that M is compact as well.

- iii. there exists $\delta > 0$ such that

$$(38) \quad M \cap (X \times [0, \delta] \times \mathbb{R}^{d-1+n}) = \partial M \times [0, \delta].$$

By the definition of $\tau_{P,m}^n$ and by the fact that $f = f_0 \wedge Id_{S^{p+m}}$, the boundary ∂M must factor as a product $\partial M = \beta M \times P$ where

$$\beta M := f_0^{-1}(G(d-p, n-m)) \subset X \times \mathbb{R}^{d-1-p+n-m}$$

is a closed submanifold.

These sub-manifolds have normal bundles given by pullbacks,

$$(39) \quad \begin{aligned} N_M &:= \tilde{f}^*(U_{d+1,n}^\perp), \\ N_{\partial M} &:= (j_d^n \circ f)^*(U_{d,n}^\perp), \\ N_{\beta M} &:= f_0^*(U_{d-p,n-m}^\perp), \end{aligned}$$

such that there is factorization

$$N_{\partial M} = N_{\beta M} \times N_P.$$

We define

$$(40) \quad \begin{aligned} T^\pi M &:= f^*(U_{d+1,n}), \\ T^\pi \partial M &:= (j_d^n \circ f)^*(U_{d,n}), \\ T^\pi \beta M &:= f_0^*(U_{d-p,n-m}). \end{aligned}$$

It follows that there are splittings,

$$(41) \quad \begin{aligned} T^\pi M|_{\partial M} &= T^\pi \partial M \oplus \epsilon^1, \\ T^\pi \partial M &= T^\pi \beta M \times TP, \end{aligned}$$

and bundle isomorphisms

$$(42) \quad \begin{aligned} N_M \oplus T^\pi M &\cong \epsilon^{d+n+1}, \\ N_{\partial M} \oplus T^\pi \partial M &\cong \epsilon^{d+n}, \\ N_{\beta M} \oplus T^\pi \beta M &\cong \epsilon^{d+n-p-m}. \end{aligned}$$

Let the map $(i_M, i_{\partial M})$ denote the inclusion of $(M, \partial M)$ into $(X \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n}, X \times \{0\} \times \mathbb{R}^{d-1+n})$, and let (π_0, π_0^∂) denote projection onto X . Clearly, $i_M^*(TX \times \epsilon^{d+n}) \cong \pi_0^*(TX) \oplus \epsilon^{d+n}$ and $i_{\partial M}^*(TX \times \epsilon^{d+n-1}) \cong (\pi_0^\partial)^*(TX) \oplus \epsilon^{d+n-1}$. This embedding yields bundle isomorphisms,

$$(43) \quad \begin{aligned} \pi_0^*(TX) \oplus \epsilon^{d+n} &\cong TM \oplus N_M \\ (\pi_0^\partial)^*(TX) \oplus \epsilon^{d+n-1} &\cong T\partial M \oplus N_{\partial M}. \end{aligned}$$

By adding $N_M, N_{\partial M}$ via Whitney-sum to both sides of the equations in (43) and using the isomorphisms of (42), we obtain a commutative diagram,

$$(44) \quad \begin{array}{ccc} TM \oplus \epsilon^{n+d+1} & \xrightarrow[\cong]{\hat{\pi}_0} & \pi_0^*TX \oplus T^\pi M \oplus \epsilon^{d+n} \\ \uparrow & & \uparrow \\ (T\partial M \oplus \epsilon^{n+d}) \oplus \epsilon^1 & \xrightarrow[\cong]{\hat{\pi}_0^\partial \oplus Id_{\epsilon^1}} & (\pi_0^*TX \oplus T^\pi \partial M \oplus \epsilon^{d+n-1}) \oplus \epsilon^1 \end{array}$$

of bundle isomorphisms where the horizontal maps cover the identity on M and the vertical maps cover the inclusion of ∂M into M . Furthermore, the bundle-map

$$\hat{\pi}_0^\partial : T\partial M \oplus \epsilon^{n+d} \longrightarrow \pi_0^*TX \oplus T^\pi\partial M \oplus \epsilon^{d+n-1}$$

has factorization:

$$(45) \quad \hat{\pi}_0^\partial = \hat{\pi}_0^\beta \times Id_{TP}$$

where,

$$\hat{\pi}_0^\beta : T\beta M \oplus \epsilon^{d+n} \longrightarrow \pi_0^*TX \oplus T^\pi\beta M \oplus \epsilon^{d+n-1}$$

is a bundle isomorphism which covers the identity on βM .

Claim 7.4. *The bundle isomorphism pair $(\hat{\pi}_0, \hat{\pi}_0^\partial \oplus Id_{\epsilon^1})$ from (44) is induced by a pair of bundle isomorphisms,*

$$\begin{array}{ccc} TM \oplus \epsilon^1 & \xrightarrow[\cong]{\hat{\pi}_1} & \pi_0^*TX \oplus T^\pi M \\ \uparrow & & \uparrow \\ (T\partial M \oplus \epsilon^1) \oplus \epsilon^1 & \xrightarrow[\cong]{\hat{\pi}_1^\partial \oplus Id_{\epsilon^1}} & (\pi_0^*TX \oplus T^\pi\partial M) \oplus \epsilon^1 \end{array}$$

with factorization

$$\hat{\pi}_1^\partial = \hat{\pi}_1^\beta \times Id_{TP}$$

where

$$\hat{\pi}_1^\beta : T\beta M \oplus \epsilon^1 \longrightarrow \pi_0^*TX \oplus T^\pi\beta M$$

is a bundle isomorphism covering the identity on βM . Furthermore, $(\hat{\pi}_1, \hat{\pi}_1^\partial \oplus Id_{\epsilon^1})$ is unique up to homotopy through bundle map pairs with the factorization specified above.

We will prove this claim in Section 11. We now define spaces

$$W := M \times \mathbb{R}, \quad \partial W := \partial M \times \mathbb{R} \quad \text{and} \quad \beta W := \beta M \times \mathbb{R}.$$

We define the bundles $T^\pi W$, $T^\pi\partial W$, $T^\pi\beta W$ to be the pullbacks of the bundles $T^\pi M$, $T^\pi\partial M$, $T^\pi\beta M$, over the projections of W , ∂W , βW onto M , ∂M , βM respectively. With W a submanifold of $X \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n}$, we set s_0 , s_0^∂ , and s_0^β to be projections of W , ∂W , and βW onto the factor X (the notation s_0 used for this projection onto X will become clear momentarily). Claim 7.4 yields bundle isomorphisms,

$$(46) \quad \begin{aligned} TW &\cong s_0^*TX \oplus T^\pi W, \\ T\partial W &\cong (s_0^\partial)^*TX \oplus T^\pi\partial W, \\ T\beta W &\cong (s_0^\beta)^*TX \oplus T^\pi\beta W, \end{aligned}$$

which cover the identity maps. Using (46) we obtain a bundle epimorphism

$$(47) \quad \begin{array}{ccc} TW & \xrightarrow{s_0} & TX \\ \downarrow & & \downarrow \\ W & \xrightarrow{s_0} & X \end{array}$$

such that the restriction $(\hat{s}_0, s_0) |_{\partial W}$ has the factorization,

$$(48) \quad \begin{array}{ccccc} T\partial W & \xrightarrow{pr} & T\beta W & \xrightarrow{\hat{s}_0^\beta} & TX \\ \downarrow & & \downarrow & & \downarrow \\ \partial W & \xrightarrow{pr} & \beta W & \xrightarrow{s_0^\beta} & X, \end{array}$$

where $(\hat{s}_0^\beta, s_0^\beta)$ is a bundle-epimorphism covering $s_0^{\beta W}$ which is projection onto X .

Claim 7.5. *There exists a homotopy (\hat{s}_t, s_t) through bundle-epimorphisms such that:*

- i. *At $t = 0$, the homotopy (\hat{s}_t, s_t) is equal to the bundle epimorphism given in (47).*
- ii. *The bundle epimorphism (\hat{s}_1, s_1) is integrable, i.e. $Ds_1 = \hat{s}_1$ and thus s_1 is a submersion.*
- iii. *For all t , (\hat{s}_t, s_t) has the factorization given in (48).*

Moreover, the integrable bundle-epimorphism (\hat{s}_1, s_1) , is unique up to homotopy through integrable bundle-epimorphisms.

We provide a proof of this claim in Section 10. This is essentially a relative version of *Phillips' Submersion Theorem* [12] adapted for P -manifolds.

Let (\hat{s}_t, s_t) be the desired family of bundle epimorphisms with the above stated properties. s_1 is now a submersion of W onto X . In order to obtain an element of $\mathbf{D}_{d+1,n}^{P,m}[X]$, we need to realize s_1 as the composition of some embedding of W into $X \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n}$ followed by projection onto X . Furthermore, we need to ensure that this new embedding of W is isotopic to our original inclusion i_W of W into $X \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n}$. Below we will construct this embedding and isotopy.

Let

$$(Id_{\mathbb{R}}, j) : \mathbb{R} \times M = W \longrightarrow \mathbb{R} \times (\mathbb{R}_+ \times \mathbb{R}^{d-1+n-p-m} \times \mathbb{R}^{p+m})$$

be the restriction to $W := \mathbb{R} \times M$ of the projection

$$X \times (\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n-p-m} \times \mathbb{R}^{p+m}) \longrightarrow \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n-p-m} \times \mathbb{R}^{p+m}$$

By rescaling the collar about ∂M , we may assume that

$$M \cap (X \times [0, 1] \times \mathbb{R}^{d-1+n-p-m} \times \mathbb{R}^{p+m}) = [0, 1] \times \partial M.$$

The map $j : M \longrightarrow \mathbb{R}_+ \times \mathbb{R}^{d-1+n-p-m} \times \mathbb{R}^{p+m}$ given above has the property that for

$$(t, b, p) \in [0, 1] \times \beta M \times P,$$

$$(49) \quad j(t, b, p) = (t, j_{\beta M}(b), i_P(p)) \in [0, 1] \times \mathbb{R}^{d-1+n-p-m} \times \mathbb{R}^{p+m}.$$

Recalling (8) we see that j is a member of the space

$$C^\infty(M, \mathbb{R}_+ \times \mathbb{R}^{d-1+n-p-m} \times \mathbb{R}^{p+m})^{(P)}.$$

The map j may fail to be an embedding. However, the map

$$\pi \times j : M \longrightarrow X \times (\mathbb{R}_+ \times \mathbb{R}^{d-1+n-p-m} \times \mathbb{R}^{p+m})$$

is an embedding; this is just the inclusion of M with π the projection onto X . Since

$$\text{Emb}(M, X \times (\mathbb{R}_+ \times \mathbb{R}^{d-1+n-p-m} \times \mathbb{R}^{p+m}))^{(P)} \subseteq C^\infty(M, X \times (\mathbb{R}_+ \times \mathbb{R}^{d-1+n-p-m} \times \mathbb{R}^{p+m}))^{(P)}$$

is an open subset, we can choose $\delta > 0$ such that if $g \in C^\infty(M, \mathbb{R}_+ \times \mathbb{R}^{d-1+n-p-m} \times \mathbb{R}^{p+m})^{(P)}$ is δ -close (under the C^∞ -norm) to j , then the map

$$\pi \times g : M \longrightarrow X \times (\mathbb{R}_+ \times \mathbb{R}^{d-1+n-p-m} \times \mathbb{R}^{p+m})$$

is an embedding. Now we choose such a δ . By Lemma C.2, assuming that $n \gg d$, there exists an embedding

$$(50) \quad e : M \longrightarrow \mathbb{R}_+ \times \mathbb{R}^{d-1+n-p-m} \times \mathbb{R}^{p+m}$$

that is δ -close to j (the precise estimate is given in C.2). We define a homotopy

$$\phi : M \times I \longrightarrow \mathbb{R}_+ \times \mathbb{R}^{d-1+n-p-m} \times \mathbb{R}^{p+m}$$

by the formula

$$\phi(x, t) = (1 - t) \cdot j(x) + t \cdot e(x).$$

Notice that for all t , the map $\phi(\cdot, t) : M \longrightarrow \mathbb{R}_+ \times \mathbb{R}^{d-1+n-p-m} \times \mathbb{R}^{p+m}$ is in

$$C^\infty(W, X \times (\mathbb{R}_+ \times \mathbb{R}^{d-1+n-p-m} \times \mathbb{R}^{p+m}))^{(P)}$$

and is δ -close to j . By how δ was chosen, this implies that the homotopy

$$\Phi : W \times I := M \times \mathbb{R} \times I \longrightarrow X \times \mathbb{R} \times (\mathbb{R}_+ \times \mathbb{R}^{d-1+n-p-m} \times \mathbb{R}^{p+m})$$

defined by the formula

$$\Phi(x, r, t) = (\pi(x), r, \phi(x, t)) \quad \text{for } (x, r, t) \in M \times \mathbb{R} \times I$$

is an isotopy through P -manifold embeddings, i.e. an isotopy through elements of

$$\text{Emb}(W, X \times \mathbb{R} \times (\mathbb{R}_+ \times \mathbb{R}^{d-1+n-p-m} \times \mathbb{R}^{p+m}))^{(P)}.$$

Recall the homotopy (\hat{s}_t, s_t) through formal submersions $W \longrightarrow X$ with (\hat{s}_1, s_1) integrable. Using s_t , Φ , and the embedding $e : M \hookrightarrow \mathbb{R}_+ \times \mathbb{R}^{d-1+n-p-m} \times \mathbb{R}^{p+m}$ from (50), we define a new isotopy

$$\Psi : W := M \times \mathbb{R} \times I \longrightarrow X \times \mathbb{R} \times (\mathbb{R}_+ \times \mathbb{R}^{d-1+n-p-m} \times \mathbb{R}^{p+m})$$

by the formula

$$\Psi(x, r, t) = \begin{cases} \Phi(x, r, 2 \cdot t) & \text{for } t \in [0, \frac{1}{2}] \\ (s_{2 \cdot t - 1}(x, r), r, e(x)) & \text{for } t \in [\frac{1}{2}, 1] \end{cases}$$

with $(x, r, t) \in M \times \mathbb{R} \times I$. This is the isotopy that we seek. The manifold

$$W' := \Psi|_{M \times \mathbb{R} \times \{1\}}(M \times \mathbb{R})$$

defines a unique element of $\mathbf{D}_{d+1,n}^{P,m}(X)$ provided that $n \gg \dim(X) + d$. Indeed, the projection onto X is a submersion by construction. Properness of the projection onto the \mathbb{R} -factor follows from the compactness of X and that of M . Here we see that our inverse map

$$H_m : [X, \Omega_{\langle P, \partial \rangle}^{\infty-1} \mathbf{MT}(d+1)_{\langle P, m, \partial \rangle}] \longrightarrow \mathbf{D}_{d+1}^{P,m}[X]$$

is well defined. One can verify directly that $T_m \circ H_m = Id$. The fact that $H_m \circ T_m = Id$ follows from Lemma 5.2. This gives the proof. \square

Lemma 7.6. *The map*

$$\sigma : \Omega_{\langle P, \partial \rangle}^{\infty-1} \mathbf{MT}(d+1)_{\langle P, m, \partial \rangle} \longrightarrow \Omega^{\infty-1} \mathbf{MT}(d+1)_{\langle P, m, \partial \rangle}$$

which was defined in (34) is a homotopy equivalence.

To prove this we define a new space that sits in between the spaces $\Omega_{\langle P, \partial \rangle}^{\infty-1} \mathbf{MT}(d+1)_{\langle P, m, \partial \rangle}$ and $\Omega^{\infty-1} \mathbf{MT}(d+1)_{\langle P, m, \partial \rangle}$.

Definition 7.2. Define $\Omega_{\langle \partial \rangle}^{d+n} \mathbf{Th}(U_{d+1,n}^\perp)_{\langle P, m, \partial \rangle}$ to be the space of pairs (\hat{f}, f) ,

$$\hat{f} \in C^0(D^{d+n}, \mathbf{Th}(U_{d+1,n}^\perp)) \quad f \in C^0(S^{d+n-1}, \mathbf{Th}(U_{d-p,n-m}^\perp \wedge S^{p+m}))$$

such that

$$\hat{f}|_{S^{d+n-1}} = \tilde{j}_d^n \circ \tau_{P,m}^n \circ f.$$

(The mapping spaces in this definition are assumed to be spaces of pointed maps.)

Notice that we do not require the map f to factor as in Definition 7.1. For each n there is an inclusion map,

$$\Omega_{\langle P, \partial \rangle}^{d+n} \mathbf{Th}(U_{d+1,n}^\perp)_{\langle P, m, \partial \rangle} \xrightarrow{\alpha_n} \Omega_{\langle \partial \rangle}^{d+n} \mathbf{Th}(U_{d+1,n}^\perp)_{\langle P, m, \partial \rangle}.$$

We take the direct limit as $n \rightarrow \infty$ to get,

$$(51) \quad \Omega_{\langle \partial \rangle}^{\infty-1} \mathbf{MT}(d+1)_{\langle P, m, \partial \rangle} := \operatorname{colim}_{n \rightarrow \infty} \Omega_{\langle \partial \rangle}^{d+n} \mathbf{Th}(U_{d+1,n}^\perp)_{\langle P, m, \partial \rangle}.$$

There is an induced map,

$$(52) \quad \Omega_{\langle P, \partial \rangle}^{\infty-1} \mathbf{MT}(d+1)_{\langle P, m, \partial \rangle} \xrightarrow{\alpha} \Omega_{\langle \partial \rangle}^{\infty-1} \mathbf{MT}(d+1)_{\langle P, m, \partial \rangle}.$$

Proposition 7.7. *The map α in (52) induces an isomorphism on homotopy groups and is thus a homotopy equivalence.*

Proof. The space $\Omega_{\langle P, \partial \rangle}^{d+n} \mathbf{Th}(U_{d+1,n}^\perp)_{\langle P, m, \partial \rangle}$ can be realized as the pullback,

$$(53) \quad \begin{array}{ccc} \Omega_{\langle P, \partial \rangle}^{d+n} \mathbf{Th}(U_{d+1,n}^\perp)_{\langle P, m, \partial \rangle} & \xrightarrow{\alpha_n} & \Omega_{\langle \partial \rangle}^{d+n} \mathbf{Th}(U_{d+1,n}^\perp)_{\langle \partial, m, \partial \rangle} \\ r \downarrow & & r \downarrow \\ \Omega_{\langle P \rangle}^{d+n-1} (\mathbf{Th}(U_{d-p,n-m}^\perp) \wedge S^{p+m}) & \xrightarrow{\alpha_n^0} & \Omega^{d+n-1} (\mathbf{Th}(U_{d-p,n-m}^\perp) \wedge S^{p+m}), \end{array}$$

where $\Omega_{\langle P \rangle}^{d+n-1}(\mathrm{Th}(U_{d-p,n-m}^\perp) \wedge S^{p+m})$ is the space of all maps of the form

$$S^{d+n-1-m-p} \wedge S^{p+m} \xrightarrow{f \wedge Id_{S^{p+m}}} \mathrm{Th}(U_{d-p,n-m}^\perp) \wedge S^{p+m},$$

the map α_n^0 is an inclusion of mapping spaces, and the two vertical maps r send functions to their restrictions. Using the fact that the Thom-Space $\mathrm{Th}(U_{d-p,n-m}^\perp)$ is $(n-m-1)$ -connected, The Freudenthal suspension Theorem implies that the suspension map

$$\pi_k(\Omega^{d-1+n-p-m} \mathrm{Th}(U_{d-p,n-m}^\perp)) \longrightarrow \pi_k(\Omega^{d-1+n} \mathrm{Th}(U_{d-p,n-m}^\perp) \wedge S^{p+m})$$

is an isomorphism when $k \leq 2(n-m-1)$. This implies that the bottom horizontal map α_0 in the diagram (53) is $2(n-m-1)$ -connected. Now, the right vertical map r is a Serre-fibration. Using the fact that the diagram is a pull-back square, hence a map of fibre-spaces, we see that α is also $2(n-m-1)$ -connected. We now just take the direct limit of these spaces as $n \rightarrow \infty$ to get our result. \square

Each element of $\Omega_{\langle \partial \rangle}^{d+n} \mathrm{Th}(U_{d+1,n}^\perp)_{\langle P,m,\partial \rangle}$ induces a map $S^{d+n} \rightarrow \mathrm{Cofibre}(\tilde{j}_d^n \circ \tau_{P,m}^n)$. Thus for each n there is a map

$$\rho_n : \Omega_{\langle \partial \rangle}^{d+n} \mathrm{Th}(U_{d+1,n}^\perp)_{\langle P,m,\partial \rangle} \longrightarrow \Omega^{d+n} \mathrm{Cofibre}(\tilde{j}_d^n \circ \tau_{P,m}^n)$$

which induces

$$\rho : \Omega_{\langle \partial \rangle}^{\infty-1} \mathbf{MT}(d+1)_{\langle P,m,\partial \rangle} \longrightarrow \Omega^{\infty-1} \mathbf{MT}(d+1)_{\langle P,m,\partial \rangle}.$$

It can be easily checked that the maps $\rho \circ \alpha$ and σ (from 34) are homotopic.

Proposition 7.8. *The above map ρ is a homotopy equivalence.*

Proof. This is similar to the last lemma. We use the following result from [8]:

Proposition 7.9. *Let (X, A) be an r -connected CW-pair, where A is s -connected such that $r, s \geq 0$. Then the map $\pi_k(X, A) \longrightarrow \pi_k(X/A)$ induced by the quotient map $X \longrightarrow X/A$ is an isomorphism for $k \leq r+s$.*

By adjunction we have an isomorphism

$$(54) \quad \pi_k(\Omega_{\langle \partial \rangle}^{d+n} \mathrm{Th}(U_{d+1,n}^\perp)_{\langle P,\partial \rangle}) \cong \pi_{k+d+n}(\mathrm{Th}(U_{d+1,n}^\perp), \mathrm{Th}(U_{d-p,n-m}^\perp) \wedge S^{p+m}).$$

By taking the mapping the mapping cylinder, we may assume that the pair

$$(\mathrm{Th}(U_{d+1,n}^\perp), \mathrm{Th}(U_{d-p,n-m}^\perp) \wedge S^{p+m})$$

is a cofibration, and the map between these spaces is an inclusion. Clearly the space, $\mathrm{Th}(U_{d-p,n-m}^\perp) \wedge S^{p+m}$ is $(n+p-1)$ -connected and the pair

$$(\mathrm{Th}(U_{d+1,n}^\perp), \mathrm{Th}(U_{d-p,n-m}^\perp) \wedge S^{p+m})$$

is at least $\min\{n-1, n+p-1\}$ -connected. We then may apply the above proposition to obtain the isomorphism,

$$\begin{aligned} \pi_{k+d+n}(\mathrm{Th}(U_{d+1,n}^\perp), \mathrm{Th}(U_{d-p,n-m}^\perp) \wedge S^{p+m}) &\cong \pi_{k+d+n}(\mathrm{Cofibre}(\tilde{j}_d^n \circ \tau_{P,m}^n)) \\ &\cong \pi_k(\Omega^{d+n} \mathrm{Cofibre}(\tilde{j}_d^n \circ \tau_{P,m}^n)). \end{aligned}$$

Taking the direct limit as $n \rightarrow \infty$ we obtain the sought after weak homotopy equivalence. \square

Combining Proposition 7.8 with Proposition 7.7 proves Lemma 7.6. Lemma 7.3 implies that the natural transformation

$$(55) \quad \mathbf{D}_{d+1}^{P,m}[X] \xrightarrow{T^m} [X, \Omega^{\infty-1} \mathbf{MT}(d+1)_{\langle P,m,\partial \rangle}] \xrightarrow{\sigma_*} [X, \Omega^{\infty-1} \mathbf{MT}(d+1)_{\langle P,m,\partial \rangle}]$$

is an isomorphism when X is compact. This proves Lemma 7.2. Note that this isomorphism does not yet imply the homotopy equivalence of $|\mathbf{D}_{d+1}^{P,m}|$ and $\Omega^{\infty-1} \mathbf{MT}(d+1)_{\langle P,m,\partial \rangle}$ in that we do not yet have a map $|\mathbf{D}_{d+1}^{P,m}| \rightarrow \Omega^{\infty-1} \mathbf{MT}(d+1)_{\langle P,m,\partial \rangle}$. In the following section, we construct such a map that will be seen to induce the isomorphism of 7.2.

7.2. A Parametrized Thom-Pontryagin Construction. We proceed in a way very similar to [11, 3.2.5]. We start with a definition:

Definition 7.3. Let $p : Y \rightarrow X$ be a submersion. Let $i_C : C \hookrightarrow Y$ be a smooth submanifold and suppose that $p|_C$ is still a submersion. A *vertical* tubular neighborhood for C in Y consists of a smooth vector bundle $q : N \rightarrow C$ (which in our case will always be the normal bundle of C) with zero section s , along with an open embedding $e : N \rightarrow Y$ such that $e \circ s = i_C$, and $p \circ e = p \circ i_C \circ q$.

Using this definition we define a variant of the sheaf $\mathbf{D}_{d+1,n}^{P,m}$ which we will denote by $\hat{\mathbf{D}}_{d+1,n}^{P,m}(X)$.

Definition 7.4. For $X \in \mathcal{Ob}(\mathcal{X})$ we define $\hat{\mathbf{D}}_{d+1,n}^{P,m}(X)$ to be the set consisting of quadruples (W, π, f, e) where $(W, \pi, f) \in \mathbf{D}_{d+1,n}^{P,m}(X)$ and

$$e : N_W \rightarrow X \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n}$$

is a vertical tubular neighborhood for W (N_W is the normal bundle for W as before) subject to the condition that the restriction of e to $N_{\partial W} = N_{\beta W} \times N_P$ is equal to the product $e_\beta \times e_P$ where

$$e_\beta : N_{\beta W} \rightarrow X \times \mathbb{R} \times \{0\} \times \mathbb{R}^{d-1+n-p-m}$$

is a vertical tubular neighborhood for βW and

$$e_P : N_P \rightarrow \mathbb{R}^{p+m}$$

is the tubular neighborhood embedding specified in (21) that was used in our construction of the spectrum $\mathbf{MT}(d+1)_{\langle P,\partial,m \rangle}$.

We define

$$(56) \quad \hat{\mathbf{D}}_{d+1}^{P,m} := \operatorname{colim}_{n \rightarrow \infty}^* \hat{\mathbf{D}}_{d+1,n}^{P,m}.$$

where like before $\operatorname{colim}_{n \rightarrow \infty}^*$ means sheaffication of $\operatorname{colim}_{n \rightarrow \infty} \hat{\mathbf{D}}_{d+1,n}^{P,m}$ which is the direct limit in presheaves. By an argument similar to the one made in Section 5, for a compact manifold X , $\operatorname{colim}_{n \rightarrow \infty} \hat{\mathbf{D}}_{d+1,n}^{P,m}(X) = \hat{\mathbf{D}}_{d+1}^{P,m}(X)$ where here the direct limit is a direct limit of sets. We also get analogues of Lemma 5.3 and Corollary 5.4. Namely, for $2k + 2d + 5 + p + m < n$ there is an isomorphism

$$\pi_k(\hat{\mathbf{D}}_{d+1,n}^{P,m}) \cong \pi_k(\hat{\mathbf{D}}_{d+1,n+1}^{P,m})$$

and there is a homotopy equivalence

$$|\hat{\mathbf{D}}_{d+1}^{P,m}| \sim \operatorname{colim}_{n \rightarrow \infty} |\hat{\mathbf{D}}_{d+1,n}^{P,m}|.$$

For each n there is a map

$$F_n : \hat{\mathbf{D}}_{d+1,n}^{P,m} \longrightarrow \mathbf{D}_{d+1,n}^{P,m}$$

defined by sending an element $(W; \pi, f, e)$ to $(W; \pi, f)$ (F_n forgets the choice of vertical tubular neighborhood embedding e). Passing to the direct limit as $n \rightarrow \infty$ yields a map of sheaves

$$F : \hat{\mathbf{D}}_{d+1}^{P,m} \longrightarrow \mathbf{D}_{d+1}^{P,m}$$

which can be seen to be a weak equivalence.

Definition 7.5. Let P, m, n , and d be as before. We define a sheaf $\mathcal{Z}_{d+1,n}^{P,m}$ on \mathcal{X} by setting

$$\mathcal{Z}_{d+1,n}^{P,m}(X) := \operatorname{Maps}(X \times \mathbb{R}, \Omega_{\langle P, \partial \rangle}^{d+n} \operatorname{Th}(U_{d+1,n}^\perp)_{\langle P, m, \partial \rangle})$$

for $X \in \mathcal{Ob}(\mathcal{X})$. Note that this is a set valued sheaf. By $\operatorname{Maps}(\cdot, \cdot)$ we mean simply the set of maps. It is not given a topology.

For each n there is map $\mathcal{Z}_{d+1,n}^{P,m} \rightarrow \mathcal{Z}_{d+1,n+1}^{P,m}$ induced by the map

$$\Omega_{\langle P, \partial \rangle}^{d+n} \operatorname{Th}(U_{d+1,n}^\perp)_{\langle P, m, \partial \rangle} \rightarrow \Omega_{\langle P, \partial \rangle}^{d+n+1} \operatorname{Th}(U_{d+1,n+1}^\perp)_{\langle P, m, \partial \rangle}$$

defined in the previous section. We define

$$\mathcal{Z}_{d+1}^{P,m} := \operatorname{colim}_{n \rightarrow \infty}^* \mathcal{Z}_{d+1,n}^{P,m}.$$

It is easy to see that $\mathcal{Z}_{d+1}^{P,m}(X) \cong \operatorname{Maps}(X \times \mathbb{R}, \Omega_{\langle P, \partial \rangle}^{\infty-1} \operatorname{MT}(d+1)_{\langle P, \partial, m \rangle})$ for any $X \in \mathcal{X}$. For each n there is a map of sheaves

$$H_n^m : \mathcal{Z}_{d+1,n}^{P,m} \longrightarrow \operatorname{Maps}(\cdot, \Omega_{\langle P, \partial \rangle}^{d+n} (\operatorname{Cofibre}(\tilde{j}_d^n \circ \tau_{P,m}^n)))$$

defined by sending a map $f : X \times \mathbb{R} \rightarrow \Omega_{\langle P, \partial \rangle}^{d+n} \operatorname{Th}(U_{d+1,n}^\perp)_{\langle P, m, \partial \rangle}$ to the composition $\sigma_n \circ f|_{X \times \{0\}}$ where

$$\sigma_n : \Omega_{\langle P, \partial \rangle}^{d+n+1} \operatorname{Th}(U_{d+1,n+1}^\perp)_{\langle P, m, \partial \rangle} \longrightarrow \operatorname{Cofibre}(\tilde{j}_d^n \circ \tau_{P,m}^n)$$

is the canonical map from (34). These maps induce a map

$$H^m : \mathcal{Z}_{d+1}^{P,m} \longrightarrow \text{Maps}(\cdot, \Omega^{\infty-1} \mathbf{MT}(d+1)_{\langle P, \partial, m \rangle}).$$

It is clear that this map is a weak equivalence of sheaves.

The Thom-Pontryagin construction yields a map of sheaves,

$$\hat{T}_n^m : \hat{\mathbf{D}}_{d+1,n}^{P,m} \longrightarrow \mathcal{Z}_{d+1,n}^{P,m}$$

which we describe in detail. Let $(W; \pi, f, e)$ be an element of $\hat{\mathbf{D}}_{d+1,n}^{P,m}(X)$. Since (π, f) is a proper map, for each $(x, t) \in X \times \mathbb{R}$ there exists a real number $\lambda(x, t) > 0$ such that if $|z| > \lambda(x, t)$ for $z \in \mathbb{R}_+ \times \mathbb{R}^{d-1+n}$, then the element $(x, t, z) \notin W$, where we are viewing (x, t, z) as an element of $X \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n}$ (λ can be taken to be a continuous function of (x, t)). From this it follows that the collapse-map induced by the vertical tubular embedding e ,

$$X \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n} \longrightarrow \text{Th}(N_W)$$

extends to

$$X \times \mathbb{R} \times D^{d-1+n} \longrightarrow \text{Th}(N_W)$$

where D^{d-1+n} is viewed as the one-point-compactification of $\mathbb{R}_+ \times \mathbb{R}^{d-1+n}$. Putting together the above Thom-Pontryagin map and the functor $\text{Th}(\cdot)$ applied to the Gauss-maps of the normal bundles we get the commutative diagram,

$$\begin{array}{ccccc} X \times \mathbb{R} \times D^{d+n} & \xrightarrow{\quad\quad\quad} & Th(N_W) & \xrightarrow{\quad\quad\quad} & \text{Th}(U_{d+1,n}^\perp) \\ \uparrow & & & & \uparrow \tilde{j}_d^n \circ \tau_{P,m}^n \\ X \times \mathbb{R} \times S^{d+n-1-p-m} \wedge S^{m+p} & \xrightarrow{\quad\quad\quad} & \text{Th}(N_{\beta W}) \wedge S^{m+p} & \xrightarrow{\quad\quad\quad} & \text{Th}(U_{d-p,n-m}^\perp) \wedge S^{m+p} \end{array}$$

which yields an element of $\mathcal{Z}_{d+1,n}^{P,m}$ via adjunction. We denote by

$$\hat{T}^m : \hat{\mathbf{D}}_{d+1}^{P,m} \longrightarrow \mathcal{Z}_{d+1}^{P,m}$$

the induced map in the direct limit as $n \rightarrow \infty$. The concordance class functors associated to our newly defined sheaves fit into the commutative diagram,

$$(57) \quad \begin{array}{ccc} \hat{\mathbf{D}}_{d+1}^{P,m}[\cdot] & \xrightarrow{\quad\quad\quad} & \mathcal{Z}_{d+1}^{P,m}[\cdot] \\ \cong \downarrow & & \cong \downarrow \\ \mathbf{D}_{d+1}^{P,m}[\cdot] & \xrightarrow{\sigma_* \circ T^m} & [\cdot, \Omega^{\infty-1} \mathbf{MT}(d+1)_{\langle P, \partial, m \rangle}] \end{array}$$

where T^m is the map defined in Section 7.1, the vertical maps are induced by F^m and H^m , and the top-horizontal map is induced by \hat{T}^m . The bottom row is an isomorphism whenever

applied to a compact manifold and so by commutativity, the top map is as well. So, for each $k \geq 0$, the map $|\hat{T}^m|$ induces a bijection

$$[S^k, |\hat{\mathbf{D}}_{d+1}^{P,m}|] \xrightarrow{\cong} [S^k, |\mathcal{Z}_{d+1}^{P,m}|].$$

We may replace $|\mathcal{Z}_{d+1}^{P,m}|$ with $\Omega^{\infty-1}\mathbf{MT}(d+1)_{\langle P, \partial, m \rangle}$ because they are homotopy equivalent via $|H^m|$. However, these are isomorphisms of sets of homotopy classes of un-based maps and not isomorphisms of the actual homotopy groups. In order to prove that $|\hat{T}^m|$ is a homotopy equivalence, we need:

Proposition 7.10. *The map $|H^m| \circ |\hat{T}^m|$ induces an isomorphism on all homotopy groups.*

Proof. The proof is similar to that of [11, 3.2.5]. We provide details in Appendix B. \square

Since the spaces involved have the homotopy type of CW complexes, the above proposition implies that $|D_{d+1}^{P,m}|$ and $\Omega^{\infty-1}\mathbf{MT}(d+1)_{\langle P, \partial, m \rangle}$ are homotopy equivalent.

8. DEPENDENCE ON THE MANIFOLD P

The construction of the spectrum $\mathbf{MT}(d+1)_{\langle P, m, \partial \rangle}$ depends on the homotopy class of the Thom-Pontryagin map

$$S^{m+p} \longrightarrow \mathbf{Th}(N_P) \longrightarrow \mathbf{Th}(U_{p+m}^\perp),$$

which is determined by the isotopy class of the chosen embedding $i_P : P \hookrightarrow \mathbb{R}^{p+m}$. If $m > p$, then any two embeddings of P into \mathbb{R}^{p+m} are isotopic. Thus, for $m_1, m_2 > p$,

$$\mathbf{MT}(d+1)_{\langle P, m_1, \partial \rangle} \sim \mathbf{MT}(d+1)_{\langle P, m_2, \partial \rangle}.$$

We are then justified in removing m from our notation and writing $\mathbf{MT}(d+1)_{\langle P, \partial \rangle}$ and $\mathbf{MT}(d+1)_{\langle P \rangle}$.

It is also interesting to note that if two manifolds P_1 and P_2 are cobordant, then their corresponding Pontryagin-Thom-collapse maps, provided that the codimension m is large enough, are homotopic. This implies that

$$\mathbf{MT}(d+1)_{\langle P_1, m, \partial \rangle} \sim \mathbf{MT}(d+1)_{\langle P_2, m, \partial \rangle}$$

provided P_1 is cobordant to P_2 and m is large. Using our main theorem we see that

$$|\mathbf{D}_{d+1}^{P_1, m}| \sim |\mathbf{D}_{d+1}^{P_2, m}|$$

and

$$\mathbf{BCob}_{d+1}^{P_1, m} \sim \mathbf{BCob}_{d+1}^{P_2, m}.$$

We summarize the above observations:

Proposition 8.1. *The homotopy types of the spectra $\mathbf{MT}(d+1)_{\langle P, \partial \rangle}$ and $\mathbf{MT}(d+1)_{\langle P \rangle}$ do not depend on the embedding $i_P : P \hookrightarrow \mathbb{R}^{p+\infty}$. Furthermore, their homotopy types are determined by a cobordism class of the manifold P .*

9. THE CLASSIFYING SPACE OF \mathbf{Cob}_{d+1}^P

In this section we prove that there is a weak homotopy equivalence,

$$|\mathbf{D}_{d+1}^{P,m}| \sim \mathbf{BCob}_{d+1}^{P,m}.$$

We first recall a fact about category valued sheaves. If \mathcal{F} is a category-valued sheaf on \mathcal{X} then the representing space $|\mathcal{F}|$ has the structure of a topological category. We have

$$\mathcal{Ob}(|\mathcal{F}|) = |N_0\mathcal{F}| \quad \text{and} \quad \mathcal{Mor}(|\mathcal{F}|) = |N_1\mathcal{F}|$$

where $N_k\mathcal{F}$ is the sheaf defined by setting $N_k\mathcal{F}(X)$ to be the k -th nerve of the category $\mathcal{F}(X)$. The classifying space $\mathbf{B}|\mathcal{F}|$ can be obtained by taking the geometric realization of the diagonal simplicial set

$$k \mapsto N_k\mathcal{F}(\Delta_e^k).$$

In this section we will define three new category-valued sheaves on \mathcal{X} :

$$\mathbf{C}_{d+1}^{P,m,\natural}, \quad \mathbf{C}_{d+1}^{P,m}, \quad \text{and} \quad \mathbf{D}_{d+1}^{P,m,\natural}.$$

We will show that there is a zig-zag of homotopy equivalences,

$$\mathbf{B}|\mathbf{C}_{d+1}^{P,m}| \xrightarrow{\sim} \mathbf{B}|\mathbf{C}_{d+1}^{P,m,\natural}| \xleftarrow{\sim} \mathbf{B}|\mathbf{D}_{d+1}^{P,m,\natural}| \xrightarrow{\sim} |\mathbf{D}_{d+1}^{P,m}|$$

and a weak homotopy equivalence

$$\mathbf{B}|\mathbf{C}_{d+1}^{P,m}| \xrightarrow{\sim} \mathbf{BCob}_{d+1}^{P,m}.$$

These equivalences together with Theorem 7.1 will yield our sought after result, Theorem 1.1. This section is very similar to what was done in [6] and [7]. The main challenge lies in giving the correct definitions of the sheaves named above. Once defined, the proofs of the equivalences are identical to the proofs given in [6] up to very slight modification.

9.1. A Note on Cocycle Sheaves. In what follows we will be using the notion of *cocycle sheaf*. For a \mathbf{CAT} -valued sheaf \mathcal{F} on \mathcal{X} , we associate a new set valued sheaf $\beta\mathcal{F}$. For a definition see [11, 4.1.1]. From the definition, it is easy to see that $\mathcal{F} \mapsto \beta\mathcal{F}$ is functorial in \mathcal{F} . This cocycle construction has the nice property that there is a homotopy equivalence

$$(58) \quad \mathbf{B}|\mathcal{F}| \sim |\beta\mathcal{F}|.$$

The proof of this is given in [11, A.3]. This homotopy equivalence is natural in the following sense. In [11, A.3] the authors construct a natural zig-zag diagram of the form,

$$(59) \quad \left(k \mapsto N_k(\mathcal{F}(\Delta_e^k)) \right) \longleftarrow \left(k \mapsto A_1^k(\mathcal{F}) \right) \longrightarrow \cdots \longleftarrow \left(k \mapsto A_l^k(\mathcal{F}) \right) \longrightarrow \left(k \mapsto \beta\mathcal{F}(\Delta_e^k) \right).$$

Each element in the above diagram is a functor from the category of **CAT**-valued sheaves on \mathcal{X} to the category of simplicial sets and each map is a natural transformation of such functors. It is easy to check that the correspondences

$$\mathcal{F} \mapsto \left(k \mapsto N_k(\mathcal{F}(\Delta_e^k)) \right) \quad \text{and} \quad \mathcal{F} \mapsto \left(k \mapsto \beta\mathcal{F}(\Delta_e^k) \right)$$

are functors in \mathcal{F} . The intermediate elements $(k \mapsto A_i^k(\mathcal{F}))$ are constructed explicitly in [11, A.3] and can be seen to be functorial in \mathcal{F} as well. The homotopy equivalence $|\mathbf{B}|\mathcal{F}| \sim |\beta\mathcal{F}|$ is proven by showing that the natural transformations induce homotopy equivalences between geometric realizations. The functoriality of this diagram will play an important role in our proof of Theorem 1.3.

9.2. The Classifying Space. For what follows, let

$$(60) \quad i_P : P \hookrightarrow \mathbb{R}^{p+m}$$

be the embedding used throughout the paper.

Notational Convention 9.1. For $X \in \mathcal{Ob}(\mathcal{X})$ and smooth functions $a, b : X \rightarrow \mathbb{R}$ with $a(x) \leq b(x)$ for all $x \in X$, we denote

$$\begin{aligned} X \times [a, b] &:= \{(x, u) \in X \times \mathbb{R} \mid a(x) < u < b(x)\}, \\ X \times (a, b) &:= \{(x, u) \in X \times \mathbb{R} \mid a(x) \leq u \leq b(x)\}. \end{aligned}$$

Definition 9.1. Let ϵ_0, ϵ_1 be fixed positive constants. Let $X \in \mathcal{Ob}(\mathcal{X})$ and $a, b : X \rightarrow \mathbb{R}$ be smooth functions with $a(x) \leq b(x)$ for all $x \in X$. We define, $\mathbf{C}_{d+1}^{P, m, \text{rh}}(X; a, b, \epsilon_0, \epsilon_1)$ to be the set of embedded submanifolds with boundary,

$$(\pi, f, j) : W \hookrightarrow X \times (a - \epsilon_0, b + \epsilon_0) \times (\mathbb{R}_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{p+m})$$

where π, f , and j are the projections onto the corresponding factors subject to the conditions:

i. The boundary,

$$\partial W = W \cap (X \times (a - \epsilon_0, b + \epsilon_0) \times \{0\} \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{p+m})$$

factors as a product $\partial W = \beta W \times P$, with $\beta W \subset X \times (a - \epsilon_0, b + \epsilon_1) \times \{0\} \times \mathbb{R}^{d-1+\infty}$ a submanifold, and $P \subset \mathbb{R}^{p+m}$ the submanifold specified in (60).

ii. The boundary ∂W is embedded with an ϵ_1 -width collar denoted by $N_{\epsilon_1}(\partial W)$ such that,

$$N_{\epsilon_1}(\partial W) = W \cap (X \times (a - \epsilon_0, b + \epsilon_0) \times [0, \epsilon_1] \times \mathbb{R}^{d-1+\infty}) = \partial W \times [0, \epsilon_1].$$

iii. The projection $\pi : W \rightarrow X$ is a submersion with $(d+1)$ -dimensional fibres. The restriction of π to the collar $N_{\epsilon_1}(\partial W)$ has the factorization,

$$N_{\epsilon_1}(\partial W) = \partial W \times [0, \epsilon_1] \xrightarrow{pr} \partial W = \beta W \times P \xrightarrow{proj} \beta_1 W \xrightarrow{\pi_{\beta W}} X$$

where the map $\pi_{\beta W}$ is a submersion with $(d-p)$ -dimensional fibres.

- iv. The map $(\pi, f) : W \rightarrow X \times (a - \epsilon_0, b + \epsilon_0)$ is proper. The restriction of f to the collar $N_{\epsilon_1}(\partial W)$ (where f is considered as a function with co-domain \mathbb{R}) has the factorization,

$$N_{\epsilon_1}(\partial W) = \partial W \times [0, \epsilon_1] \xrightarrow{pr} \partial W = \beta W \times P \xrightarrow{pr} \beta_1 W \xrightarrow{f_{\beta W}} \mathbb{R},$$

where the map $(\pi_{\beta W}, f_{\beta W})$, is proper as well.

- v. The restrictions of (π, f) to $(\pi, f)^{-1}(X \times (a - \epsilon, a + \epsilon))$ and of $(\pi_{\beta W}, f_{\beta W})$ to $(\pi_{\beta W}, f_{\beta W})^{-1}(X \times (a - \epsilon, a + \epsilon))$ are submersions. Same holds for b .

We eliminate dependence on the ϵ_0, ϵ_1 by setting

$$(61) \quad \mathbf{C}_{d+1}^{P,m,\natural}(X; a, b) := \operatorname{colim}_{\epsilon_0, \epsilon_1 \rightarrow 0} \mathbf{C}_{d+1}^{P,m,\natural}(X; a, b, \epsilon_0, \epsilon_1).$$

Definition 9.2. Let $\mathbf{C}_{d+1}^{P,m,\natural}(X) := \bigsqcup \mathbf{C}_{d+1}^{P,m,\natural}(X; a, b)$ with union ranging over all pairs of smooth real-valued functions (a, b) with $a \leq b$.

This definition makes $\mathbf{C}_{d+1}^{P,m,\natural}(X)$ into a category-valued sheaf. This sheaf should be compared to $\mathbf{C}_{d+1}^{\natural}$ defined in [6]. Indeed they are equal in the case that $P = \emptyset$. The composition is defined by gluing manifolds along common ∂_0 -components.

Definition 9.3. Let $\mathbf{C}_{d+1}^{P,m}(X, a, b, \epsilon_0, \epsilon_1) \subset \mathbf{C}_{d+1}^{P,m,\natural}(X, a, b, \epsilon_0, \epsilon_1)$ be the subset satisfying the further condition:

- vi. For $x \in X$ let J_a be the interval $((a - \epsilon_0)(x), (a + \epsilon_0)(x)) \subseteq \mathbb{R}$ and let

$$V_a = (\pi, f)^{-1}(\{x\} \times J_a) \subseteq \{x\} \times J_a \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{p+m}.$$

Then

$$V_a = \{x\} \times J_a \times M \subseteq \{x\} \times J_a \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{p+m}$$

for some $(d - 1)$ -dimensional submanifold M with boundary. The same condition must hold for the function b .

It follows that the boundary $\partial M = M \cap (\{x\} \times J_a \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{p+m})$ has the factorization $\partial M = \beta M \times P$.

We then may define the sheaf $\mathbf{C}_{d+1}^{P,m}$ in the same way as done earlier by taking the limits as $\epsilon_0, \epsilon_1 \rightarrow 0$ and taking the disjoint union over all pairs of real valued functions $a \leq b$.

It can be checked that $\mathbf{C}_{d+1}^{P,m} \cong \mathbf{C}^\infty(\cdot, \mathbf{Cob}_{d+1}^{P,m})$. Using this identification there is a canonical map

$$(62) \quad \eta : |\mathbf{C}_{d+1}^{P,m}| \longrightarrow \mathbf{Cob}_{d+1}^{P,m}$$

which induces

$$\mathbf{B}(\eta) : \mathbf{B}|\mathbf{C}_{d+1}^{P,m}| \longrightarrow \mathbf{BCob}_{d+1}^{P,m}.$$

We have:

Proposition 9.1. *The above map $\mathbf{B}(\eta)$ is a weak homotopy equivalence.*

Proof. This follows directly from [6, Proposition 2.9]. \square

Proposition 9.2. *The map $\mathbf{C}_{d+1}^{P,m} \rightarrow \mathbf{C}_{d+1}^{P,m,\natural}$ given by inclusion induces a weak homotopy equivalence,*

$$\mathbf{B}|\mathbf{C}_{d+1}^{P,m}| \xrightarrow{\sim} \mathbf{B}|\mathbf{C}_{d+1}^{P,m,\natural}|.$$

Proof. This follows directly from [6, Proposition 4.4]. \square

We now define a new sheaf which we can compare to $\mathbf{D}_{d+1}^{P,m}$.

Definition 9.4. We define $\mathbf{D}_{d+1}^{P,m,\natural}(X)$ to be the set of pairs (W, a) such that,

- i. $W \in \mathbf{D}_{d+1}^{P,m}(X)$,
- ii. $a : X \rightarrow \mathbb{R}$ is a smooth function,
- iii. the function $f : W \rightarrow \mathbb{R}$ and the restriction $f|_{\partial W}$, from the definition of $\mathbf{D}_{d+1}^{P,m}(X)$, are fibrewise transverse to a . This means that for any $x \in X$, the restriction of f to the fibre $\pi^{-1}(x)$ is transverse to $a(x)$.

The sheaf $\mathbf{D}_{d+1}^{P,m,\natural}$ is then a category valued sheaf. A morphism of $\mathbf{D}_{d+1}^{P,m,\natural}(X)$ is given by a triple (W, a, b) , where (W, a) and (W, b) are elements of $\mathbf{D}_{d+1}^{P,m,\natural}$ and $a(x) \leq b(x)$ for all $x \in X$. This definition should be compared to the sheaf $\mathbf{D}_{d+1}^{\natural}(X)$ defined in [6]. Indeed, $\mathbf{D}_{d+1}^{P,m,\natural} \cong \mathbf{D}_{d+1}^{\natural}$ in the case that $P = \emptyset$.

We now consider the cocycle sheaf $\beta\mathbf{D}_{d+1}^{P,m,\natural}$. There is a “forgetful map”,

$$(63) \quad \beta\mathbf{D}_{d+1}^{P,m,\natural} \xrightarrow{h} \mathbf{D}_{d+1}^{P,m}.$$

Proposition 9.3. *The above “forgetful map” h is a weak equivalence of sheaves.*

Proof. This follows directly from [6, Proposition 4.2]. \square

We now consider a map of sheaves,

$$(64) \quad \mathbf{D}_{d+1}^{P,m,\natural} \xrightarrow{\alpha} \mathbf{C}_{d+1}^{P,m,\natural}$$

given as follows:

By properness of (π, f) , there exists a smooth function $\epsilon_0 : X \rightarrow (0, \infty)$ such that (π, f) and $(\pi|_{\partial W}, f|_{\partial W})$ restricted to the open subsets

$$W_{\epsilon_0} = (\pi, f)^{-1}(X \times (a - \epsilon_0, a + \epsilon_0)) \quad \text{and} \quad \partial W_{\epsilon_0} = (\pi|_{\partial W}, f|_{\partial W})^{-1}(X \times (a - \epsilon_0, a + \epsilon_0))$$

are proper submersions. It follows that $(W_{\epsilon_0}, a, a) \in \text{Ob}(\mathbf{C}_{d+1}^{P,m,\natural}(X))$

So for an object $(W, a) \in \mathcal{O}b(\mathbf{D}_{d+1}^{P,m,\natural}(X))$ we define $\alpha((W, a)) := (W_{\epsilon_0}, a, a)$. We define α on morphisms similarly.

Proposition 9.4. *The map α defined above induces a weak homotopy equivalence*

$$\mathbf{B}|\mathbf{D}_{d+1}^{P,m,\natural}| \sim \mathbf{B}|\mathbf{C}_{d+1}^{P,m,\natural}|.$$

Proof. Follows directly from [6, Proposition 4.4]. \square

The last four propositions imply that there is a weak homotopy equivalence $|\mathbf{D}_{d+1}^{P,m}| \sim \mathbf{BCob}_{d+1}^{P,m}$. Combining this with Theorem 7.1 proves 1.1.

9.3. The Boundary Map. We now consider the functor $\beta_1 : \mathbf{Cob}_{d+1}^{P,m} \longrightarrow \mathbf{Cob}_{d-p}$ defined by sending a P -manifold W to $\beta_1 W$. Theorem 1.3 stated in the introduction asserts that the homotopy fibre of $\mathbf{B}(\beta_1) : \mathbf{BCob}_{d+1}^{P,m} \longrightarrow \mathbf{BCob}_{d-p}$ has the homotopy type of \mathbf{BCob}_{d+1} . We give a proof of this assertion.

Proof of Theorem 1.3. The functor $\beta_1 : \mathbf{Cob}_{d+1}^{P,m} \longrightarrow \mathbf{Cob}_{d-p}$ induces a map of sheaves

$$\beta_1 : \mathbf{C}_{d+1}^{P,m} \longrightarrow \mathbf{C}_{d-p}.$$

This yields a diagram

$$\begin{array}{ccc} \mathbf{B}|\mathbf{C}_{d+1}^{P,m}| & \xrightarrow[\sim]{\mathbf{B}(\eta)} & \mathbf{BCob}_{d+1}^{P,m} \\ \downarrow \mathbf{B}|\beta_1| & & \downarrow \mathbf{B}(\beta_1) \\ \mathbf{B}|\mathbf{C}_{d-p}| & \xrightarrow[\sim]{\mathbf{B}(\eta)} & \mathbf{BCob}_{d-p} \end{array}$$

where the horizontal maps are the one from (62) where the bottom horizontal map applied to the case $P = \emptyset$. This diagram commutes by the naturality of the map η from (62). Similarly, we can define maps of sheaves

$$\begin{aligned} \beta_1^\natural : \mathbf{C}_{d+1}^{P,m,\natural} &\longrightarrow \mathbf{C}_{d-p}^\natural \\ \beta_1^\natural : \mathbf{D}_{d+1}^{P,m,\natural} &\longrightarrow \mathbf{D}_{d-p}^\natural \end{aligned}$$

making the diagram

$$(65) \quad \begin{array}{ccccc} \mathbf{C}_{d+1}^{P,m} & \xrightarrow{i} & \mathbf{C}_{d+1}^{P,m,\natural} & \xleftarrow{\alpha} & \mathbf{D}_{d+1}^{P,m,\natural} \\ \downarrow \beta_1 & & \downarrow \beta_1^\natural & & \downarrow \beta_1^\natural \\ \mathbf{C}_{d-p} & \xrightarrow{i} & \mathbf{C}_{d-p}^\natural & \xleftarrow{\alpha} & \mathbf{D}_{d-p}^\natural \end{array}$$

commute. This leads to the commutative zig-zag diagram,

$$(66) \quad \begin{array}{ccccccc} \mathbf{BCob}_{d+1}^{P,m} & \xleftarrow{\sim} & \mathbf{B}|\mathbf{C}_{d+1}^{P,m}| & \xrightarrow{\sim} & \mathbf{B}|\mathbf{C}_{d+1}^{P,m,\natural}| & \xleftarrow{\sim} & \mathbf{B}|\mathbf{D}_{d+1}^{P,m,\natural}| \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbf{BCob}_{d-p} & \xleftarrow{\sim} & \mathbf{B}|\mathbf{C}_{d-p}| & \xrightarrow{\sim} & \mathbf{B}|\mathbf{C}_{d-p}^\natural| & \xleftarrow{\sim} & \mathbf{B}|\mathbf{D}_{d-p}^\natural|. \end{array}$$

It follows that the vertical maps in the above diagram all have weakly equivalent homotopy-fibres.

Now, the map β^\natural induces a map of cocycle-sheaves,

$$\beta(\beta^\natural) : \beta \mathbf{D}_{d+1}^{P,m,\natural} \longrightarrow \beta \mathbf{D}_{d-p}^\natural$$

such that the diagram

$$\begin{array}{ccc} \beta \mathbf{D}_{d+1}^{P,m,\natural} & \xrightarrow{h} & \mathbf{D}_{d+1}^{P,m} \\ \downarrow \beta(\beta^\natural) & & \downarrow \beta \\ \beta \mathbf{D}_{d-p}^\natural & \xrightarrow{h} & \mathbf{D}_{d-p} \end{array}$$

commutes, where h is the forgetful map from (63). (This clash of the notation here is unfortunate. However, the use of the symbol β for both cocycle sheaves and for P -manifolds is standard. This is the only place in which we will have to deal with such notational issues.) The functoriality of the zig-zag diagram (9.1) implies that the diagram

$$\begin{array}{ccccc} |\beta \mathbf{D}_{d+1}^{P,m,\natural}| & \xrightarrow{\sim} & \dots & \xleftarrow{\sim} & \mathbf{B}|\mathbf{D}_{d+1}^{P,m,\natural}| \\ \downarrow |\beta(\beta^\natural)| & & & & \downarrow \mathbf{B}|\beta^\natural| \\ |\beta \mathbf{D}_{d-p}^\natural| & \xrightarrow{\sim} & \dots & \xleftarrow{\sim} & \mathbf{B}|\mathbf{D}_{d-p}^\natural| \end{array}$$

commutes where the horizontal series of maps comes from the zig-zag (9.1). There is also the commutative diagram,

$$\begin{array}{ccc} |\beta \mathbf{D}_{d+1}^{P,m,\natural}| & \xrightarrow[\sim]{|\mathbf{h}|} & |\mathbf{D}_{d+1}^{P,m}| \\ \downarrow \mathbf{B}|\beta^\natural| & & \downarrow |\beta| \\ |\beta \mathbf{D}_{d-p}^\natural| & \xrightarrow[\sim]{\mathbf{B}|h|} & |\mathbf{D}_{d-p}|, \end{array}$$

which implies that the homotopy fibre of right vertical map has same weak homotopy type as $\mathbf{B}(\beta_1) : \mathbf{BCob}_{d+1}^P \longrightarrow \mathbf{BCob}_{d-p}$.

We can define a similar map

$$\hat{\beta}^n : \hat{\mathbf{D}}_{d+1,n}^{P,m} \longrightarrow \hat{\mathbf{D}}_{d-p,n}$$

which sends an element $(W; \pi, f, e)$ to $(W; \pi_{\beta W}, f_{\beta W}, e_\beta)$ which induces

$$\hat{\beta} : \hat{\mathbf{D}}_{d+1}^{P,m} \longrightarrow \hat{\mathbf{D}}_{d-p}.$$

Passing to representing spaces we obtain a diagram,

$$\begin{array}{ccccc} |\mathbf{D}_{d+1}^{P,m}| & \xleftarrow{\sim} & |\hat{\mathbf{D}}_{d+1}^{P,m}| & \xrightarrow[\sim]{|H^m| \circ |\hat{T}^m|} & \Omega^{\infty-1} \mathbf{MT}(d+1)_{\langle P, \partial, m \rangle} \\ \downarrow |\beta| & & \downarrow |\hat{\beta}| & & \downarrow r \\ |\mathbf{D}_{d-p}| & \xleftarrow{\sim} & |\hat{\mathbf{D}}_{d-p}| & \xrightarrow[\sim]{|H^m| \circ |\hat{T}^m|} & \Omega^{\infty-1} \mathbf{MT}(d-p) \end{array}$$

for which the left square commutes and the right square can be seen to commute up to homotopy. The horizontal homotopy equivalences imply that the homotopy fibres of the vertical maps are homotopy equivalent. Putting this together with (66) implies that the homotopy-fibres of the maps

$$\begin{aligned} \mathbf{B}(\beta_1) : \mathbf{BCob}_{d+1}^P &\longrightarrow \mathbf{BCob}_{d-p}, \\ r : \Omega^{\infty-1}\mathbf{MT}(d+1)_{\langle P, \partial \rangle} &\longrightarrow \Omega^{\infty-1}\mathbf{MT}(d-p) \end{aligned}$$

have the same weak homotopy type. The weak homotopy equivalence

$$\mathbf{BCob}_{d+1} \sim \Omega^{\infty-1}\mathbf{MT}(d+1)$$

implies that the homotopy fibre of

$$\mathbf{B}(\beta_1) : \mathbf{BCob}_{d+1}^P \longrightarrow \mathbf{BCob}_{d-p}$$

is weakly equivalent to \mathbf{BCob}_{d+1} . □

10. THE SUBMERSION THEOREM

In this section we prove a result which implies Claim 7.5 used in the proof of 7.3. Our result is a relative version of *Phillips' Submersion Theorem* [12] adapted for P -manifolds.

Definition 10.1. We define the space $\text{Sub}_{\beta W, P}(W, V)$ to be the space of all submersions $f : W \longrightarrow V$ with the property that the restriction $f|_{\partial W}$ is also a submersion and of the form

$$\partial W = \beta W \times P \xrightarrow{\text{proj}_{\beta W}} \beta W \xrightarrow{f_{\beta W}} V$$

It follows by definition that $f_{\beta W}$ is also a submersion.

Definition 10.2. Define the space $\text{Sur}_{\beta W, P}(TW, TX)$ to be the space of formal submersions (fiberwise surjective bundle maps),

$$\begin{array}{ccc} TW & \xrightarrow{\hat{f}} & TX \\ \downarrow & & \downarrow \\ W & \xrightarrow{f} & X \end{array}$$

such that $(\hat{f}, f)|_{\partial W}$ has the following properties:

- (1) $f|_{\partial W}$ factors in the same way defined above for the space of submersions.
- (2) $\hat{f}|_{\partial W}$ is surjective on fibres and is of the form $\hat{f}|_{\partial W} = \hat{f}_{\beta W} \times \text{id}_{TP}$.

Theorem 10.1. *Let W be an open manifold with the property that $\beta W \times P = \partial W$ is also open (i.e. every connected component has an escape to infinity). Then the map*

$$\text{Sub}_{\beta W, P}(W, V) \xrightarrow{D} \text{Sur}_{\beta W, P}(TW, TX)$$

given by the map

$$f \longmapsto (f, Df)$$

is a weak homotopy equivalence.

For definitions and basic results regarding open manifolds see [5, 4.3].

Proof. We give ∂W an ϵ -width collar-neighborhood which we denote by $N_\epsilon(\partial W)$. This collar neighborhood is of course diffeomorphic to the product of ∂W with an interval and so there is a projection map

$$N_\epsilon(\partial W) \longrightarrow \partial W = \beta W \times P \longrightarrow \beta W$$

which we denote by pr_ϵ . We define the space

$$\text{Sub}_{\beta W, P}^\epsilon(W, V) \subset \text{Sub}_{\beta W, P}(W, V)$$

to be the subspace consisting of all submersions f such that $f|_{N_\epsilon(\partial W)}$ factors as such,

$$N_\epsilon(\partial W) \xrightarrow{\text{proj}} \partial W = \beta W \times P \xrightarrow{\text{proj}} \beta W \xrightarrow{f_{\beta W}} V.$$

The space $\text{Sur}_{\beta W, P}^\epsilon(W, V)$ is defined similarly.

Now, the spaces $\text{Sub}_{\beta W, P}^\epsilon(W, V)$ and $\text{Sur}_{\beta W, P}^\epsilon(W, V)$ fit into pull-back diagrams

$$(67) \quad \begin{array}{ccc} \text{Sub}_{\beta W, P}^\epsilon(W, V) & \longrightarrow & \text{Sub}(W, V) \\ \downarrow r_{\epsilon, \beta} & & \downarrow r \\ \text{Sub}(\beta W, V) & \xrightarrow{pr_\epsilon^*} & \text{Sub}(N_\epsilon(\partial W), V) \end{array} \quad \begin{array}{ccc} \text{Sur}_{\beta W, P}^\epsilon(W, V) & \longrightarrow & \text{Sur}(W, V) \\ \downarrow r_{\epsilon, \beta} & & \downarrow r' \\ \text{Sur}(\beta W, V) & \xrightarrow{pr_\epsilon^*} & \text{Sur}(N_\epsilon(\partial W), V) \end{array}$$

The right vertical maps r and r' are defined by $f \mapsto f|_{N_\epsilon(\partial W)}$ and the bottom horizontal maps in both diagrams are induced by the projections $pr_\epsilon : N_\epsilon(\partial W) \rightarrow \beta W$. We claim that the restriction maps

$$r : \text{Sub}(W, V) \longrightarrow \text{Sub}(N_\epsilon(\partial W), V)$$

$$r' : \text{Sur}(W, V) \longrightarrow \text{Sur}(N_\epsilon(\partial W), V)$$

are Serre-fibrations. We will prove this next. Since r and r' are Serre-fibrations, the spaces $\text{Sub}_{\beta W, P}^\epsilon(W, V)$ and $\text{Sur}_{\beta W, P}^\epsilon(W, V)$ are actually the homotopy pull-backs of the diagrams (67). Now, by Phillip's submersion theorem [12], the maps

$$\begin{aligned} \text{Sub}(\beta W, V) &\xrightarrow{D} \text{Sur}(\beta W, V), \\ \text{Sub}(N_\epsilon(\partial W), V) &\xrightarrow{D} \text{Sur}(N_\epsilon(\partial W), V), \\ \text{Sub}(W, V) &\xrightarrow{D} \text{Sur}(W, V), \end{aligned}$$

are all weak-homotopy equivalences compatible with the diagrams of 67. It follows that

$$\text{Sub}_{\beta W, P}^\epsilon(W, V) \xrightarrow{D} \text{Sur}_{\beta W, P}^\epsilon(W, V)$$

is a homotopy equivalence. Taking the direct limit as $\epsilon \rightarrow 0$ we get that

$$\text{Sub}_{\beta W, P}(W, V) \xrightarrow{D} \text{Sur}_{\beta W, P}(W, V)$$

is a weak homotopy equivalence. □

Lemma 10.2. *The restriction map*

$$r : \text{Sub}(W, V) \longrightarrow \text{Sub}(N_\epsilon(\partial W), V)$$

is a fibration.

Proof. To prove this we will first need to prove that $r' : \text{Sur}(W, V) \longrightarrow \text{Sur}(N_\epsilon(\partial W), V)$ is a Serre-fibration. The space $\text{Sur}(W, V)$ can be identified with the space of sections of the locally trivial fibre-bundle given by the composition

$$\text{Hom}_{\text{Max Rank}}(\pi_W^*(TW), \pi_V^*(TV)) \longrightarrow W \times V \xrightarrow{\pi_W} W .$$

The proof that $r' : \text{Sur}(W, V) \longrightarrow \text{Sur}(N_\epsilon(\partial W), V)$ is a Serre-fibration comes from the following:

Proposition 10.3. *Let $E \longrightarrow B$ be a locally trivial fibre-bundle with B is a Manifold. Let A be a submanifold. Then the restriction map*

$$\Gamma_B(E) \longrightarrow \Gamma_A(E|_A)$$

is a Serre-fibration.

Proof. For $E \longrightarrow B$ a trivial fibre-bundle, the proof follows from the well known fact that the restriction map:

$$\mathcal{C}^0(B, F) \longrightarrow \mathcal{C}^0(A, F)$$

is a fibration when the pair (B, A) is a cofibration, F is any space. To extend this result to any locally trivial fibre-bundle, choose a triangulation of B that restricts to a triangulation of A . Using the fact that the bundle E is trivialized over any simplex, one can proceed by an induction on the simplices of B . □

Now to show that the restriction map

$$\text{Sub}(W, V) \longrightarrow \text{Sub}(N_\epsilon(\partial W), V)$$

is a fibration. Let X be a finite CW complex. Consider the diagram

$$\begin{array}{ccccc} X \times \{0\} & \xrightarrow{\tilde{f}} & \text{Sub}(W, V) & \xrightarrow{D} & \text{Sur}(TW, TV) \\ \downarrow & & \downarrow & & \downarrow \\ X \times [0, 1] & \xrightarrow{F} & \text{Sub}(N_\epsilon(\partial W), V) & \xrightarrow{D} & \text{Sur}(TN_\epsilon(\partial W), TV). \end{array}$$

Proposition 10.3 implies that $D \circ F$ lifts to $(D \circ F)' : X \times [0, 1] \longrightarrow \text{Sur}(TW, TV)$ such that $(D \circ F)'|_{W \times \{0\}} = D \circ \tilde{f}$. Now we can apply the *Relative-Parametric H-principle* from [5] to get a homotopy through formal submersions s_t , such that

- i. $s_0 = (D \circ F)'$,
- ii. s_1 is integrable and hence represented by an actual submersion.
- iii. $s_t|_{X \times \{0\}} = \tilde{f}$ for all t ,
- iv. $s_t(x, r)|_{T\partial W \times [0, \epsilon]} = F(x, r)$ for all $(x, r) \in X \times [0, 1]$ and for all t .

With this family of formal submersions constructed, we see that s_1 gives us the desired lift of the homotopy F proving that our restriction map is indeed a Serre-fibration. \square

11. THE STABILIZATION MAP

In this section we prove a lemma that implies Claim 7.4 used in the proof of Lemma 7.3. This result is essentially a relative version of [11, Lemma 3.2.3].

For any space X and vector bundles V_1 , and V_2 over X , let $\text{Iso}_X(V_1, V_2)$ be the space of bundle-isomorphisms covering Id_X . There is a stabilization map

$$(68) \quad \begin{aligned} \sigma : \text{Iso}_X(V_1, V_2) &\longrightarrow \text{Iso}_X(V_1 \oplus \epsilon^1, V_2 \oplus \epsilon^1) \quad \text{given by} \\ f &\mapsto f \oplus \text{Id}_{\epsilon^1}. \end{aligned}$$

For what follows, let M be a compact P -manifold with $\partial_0 M = \emptyset$ (in this case like in the previous sections we will skip indices on ∂ and β since there is no ambiguity).

Let $h : \partial M \times [0, 1) \rightarrow M$ be a collar embedding and set $N_\delta(\partial M) := h(\partial M \times [0, \delta])$ where $0 < \delta \leq 1$. Let E^1, E^2 be vector bundles of the same fibre-dimension over M such that their restrictions to $N_\delta(\partial M)$ have splittings,

$$(69) \quad \begin{aligned} E^i|_{N_\delta(\partial M)} &\cong \hat{E}^i \times \epsilon^1, \\ \hat{E}^i &= E_{\beta M}^i \times E_P, \end{aligned}$$

where $\hat{E}^i \rightarrow \partial M$, $E_{\beta M}^i \rightarrow \beta M$, and $E_P \rightarrow P$ are vector bundles, and $i = 1, 2$.

Definition 11.1. Let E^1, E^2 be vector bundles over M with the splittings specified in (69). We define $\text{Iso}_M^{\beta, P}(E^1, E^2)$ to be the space of bundle isomorphisms $f : E^1 \rightarrow E^2$ covering Id_M , that satisfy the following conditions:

- i. The restriction of f to $N_\delta(\partial M)$ has splitting,

$$f|_{\partial M} = \hat{f} \times \text{Id}_{\epsilon^1}$$

where $\hat{f} : \hat{E}_1 \rightarrow \hat{E}_2$ is a bundle isomorphism covering $\text{Id}_{\partial M}$.

- ii. The bundle map $\hat{f} : \hat{E}_1 \rightarrow \hat{E}_2$ has splitting,

$$\hat{f} = f_{\beta M} \times \text{Id}_{E_P}$$

where $f_{\beta M} : E_{\beta M}^1 \rightarrow E_{\beta M}^2$ is a bundle isomorphism covering $\text{Id}_{\beta M}$.

Just like in (68), there is a stabilization map

$$(70) \quad \text{Iso}_M^{\beta,P}(E^1, E^2) \xrightarrow{\sigma} \text{Iso}_M^{\beta,P}(E^1 \oplus \epsilon^1, E^2 \oplus \epsilon^1),$$

given by

$$f \longmapsto f \oplus \text{Id}_{\epsilon^1}.$$

Theorem 11.1. *Let k be the fibre-dimension of E^i such that $k > \dim(M)$. Let p be the fibre-dimension of E_P and so $k - p - 1$ is the fibre-dimension of $E_{\beta M}^i$. The stabilization map σ from (70) is $(k - \dim(M) - 1)$ -connected.*

Proof. First note that the space $\text{Iso}_M^{\beta,P}(E^1, E^2)$ can be realized as the pull-back of the diagram,

$$(71) \quad \begin{array}{ccc} & \text{Iso}_M(E^1, E^2) & \\ & \downarrow & \\ \text{Iso}_{\beta M}(E_{\beta M}^1, E_{\beta M}^2) & \longrightarrow & \text{Iso}_{N_\delta(\partial M)}(E^1|_{N_\delta(\partial M)}, E^2|_{N_\delta(\partial M)}) \end{array}$$

where the right-vertical map is given by restriction, and the bottom horizontal map is given by

$$f \longmapsto f \times \text{Id}_{E_P} \times \text{Id}_{\epsilon^1}.$$

The vertical map in (71) is a Serre-fibration. This implies that $\text{Iso}_M^{\beta,P}(E^1, E^2)$ is in fact homotopy equivalent to the homotopy pull-back of (71) and is homotopy invariant. We now apply [11, Lemma 3.2.3] to see that the stabilization maps,

$$\text{Iso}(E^1, E^2) \xrightarrow{\sigma} \text{Iso}(E^1 \oplus \epsilon, E^2 \oplus \epsilon),$$

$$\text{Iso}_{\partial M}(E^1|_{\partial M}, E^2|_{\partial M}) \xrightarrow{\sigma} \text{Iso}_{\partial M}(E^1|_{\partial M} \oplus \epsilon, E^2|_{\partial M} \oplus \epsilon),$$

$$\text{Iso}_{\beta M}(E_{\beta M}^1, E_{\beta M}^2) \xrightarrow{\sigma} \text{Iso}_{\beta M}(E_{\beta M}^1 \oplus \epsilon, E_{\beta M}^2 \oplus \epsilon),$$

are each $(k - \dim(M) - 1)$ -connected. These stabilization maps are natural with respect to the pull-back diagram (71). Using the fact that the vertical maps in the pull-back diagram are Serre fibrations, the stabilization map

$$\sigma : \text{Iso}_M^{\beta,P}(E^1, E^2) \longrightarrow \text{Iso}_M^{\beta,P}(E^1 \oplus \epsilon^1, E^2 \oplus \epsilon^1)$$

is $(k - \dim(M) - 1)$ -connected. □

APPENDIX A.

In this section we prove two lemmas which imply Lemma 3.2 and Theorem 3.3. These results are slight modifications of results from [3] and [10]. The proofs are very similar to the ones given in these papers.

Lemma A.1. *Let W be a d -dimensional P -manifold. Let ϵ_0, ϵ_1 be fixed positive constants. Fix an embedding $P \hookrightarrow \mathbb{R}^{p+m}$. The quotient map,*

$$q : \text{Emb}(W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n})_{\epsilon_0, \epsilon_1}^{\langle P \rangle} \longrightarrow \frac{\text{Emb}(W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n})_{\epsilon_0, \epsilon_1}^{\langle P \rangle}}{\text{Diff}(W)_{\epsilon_0, \epsilon_1}^{\langle P \rangle}}$$

is a locally trivial fibre bundle.

Proof. Let $f \in \text{Emb}(W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n})_{\epsilon_0, \epsilon_1}^{\langle P \rangle}$ and let $[f]$ denote the class of f in

$$\frac{\text{Emb}(W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n})_{\epsilon_0, \epsilon_1}^{\langle P \rangle}}{\text{Diff}(W)_{\epsilon_0, \epsilon_1}^{\langle P \rangle}}.$$

Notice that $\hat{f}(W)$ is the same submanifold for all $\hat{f} \in q^{-1}([f])$. We will refer to this submanifold as $f(W)$. Let $N \rightarrow f(W)$ be the normal bundle for $f(W)$ in $[0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n}$. We choose a tubular neighborhood \tilde{N} of $f(W)$ in the standard way by taking \tilde{N} to be the image of the exponential map

$$\exp : N \longrightarrow [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n}$$

restricted to a sufficiently small neighborhood of the zero section N . By compactness of W we may take this sufficiently small neighborhood of the zero section to be

$$N_\delta = \{(x, v) \in N \mid |v| < \delta\}$$

where δ is a sufficiently small positive real number. Recall that \exp is defined by sending $(x, v) \in N$ to the point $\gamma_{x,v}(1)$ where $\gamma_{x,v}$ is the unique geodesic segment starting at x and with initial velocity equal to v . There is a projection map

$$\pi : \tilde{N} \longrightarrow f(W)$$

defined to be the inverse of $\exp|_{N_\delta}$. Notice that since we are in Euclidean space, these geodesic segments are just straight lines orthogonal $f(W)$.

Claim A.2. *Let $(s, x, p) \in [0, \epsilon_1] \times ([0, 1] \times \mathbb{R}^{d-1+n-p-m}) \times i_P(P)$. We then have*

$$(72) \quad \pi(s, x, p) = (s, \pi^{\beta_1}(x), p)$$

where π^{β_1} is the projection map for a tubular neighborhood of $f(\beta_1 W)$ in the product

$$\{0\} \times [0, 1] \times \mathbb{R}^{d-1+n-p-m}.$$

Proof of Claim. Notice that the restriction of the normal bundle N to the submanifold

$$[0, \epsilon_1] \times f(\beta_1 W) \times i_P(P) = [0, \epsilon_1] \times f(\partial_1 W)$$

has the factorization $N|_{[0, \epsilon_1] \times f(\beta_1 W) \times i_P(P)} = \{0\} \times N_{\beta_1 W} \times N_P$, where $N_{\beta_1 W}$ is the normal bundle for $\beta_1 W$ in $[0, 1] \times \mathbb{R}^{d-1+n-p-m}$ and N_P is the normal bundle for $i_P(P)$ in \mathbb{R}^{m+p} . From this, it follows that for any point $(s, x, p) \in [0, \epsilon_1] \times f(\beta_1 W) \times i_P(P)$ any geodesic segment starting at (s, x, p) and with initial velocity normal to $[0, \epsilon_1] \times f(\beta_1 W) \times i_P(P)$ will have the form $t \mapsto (s, \gamma_x^{\beta_1}(t), \gamma_p^P(t))$ where $\gamma_x^{\beta_1}$ is a geodesic in $[0, 1] \times \mathbb{R}^{d-1+n-p-m}$ starting at x with initial velocity normal to $\beta_1 W$ and γ_p^P is a geodesic in \mathbb{R}^{p+m} starting at p with initial velocity normal to $i_P(P)$. The claim follows easily from these observations. \square

Now let \tilde{N} be a tubular neighborhood of $f(W)$ chosen as above. \tilde{N} has the property that if $\text{dist}(x, f(W)) < \delta$ then $x \in \tilde{N}$. Now let

$$U \subset \text{Emb}(W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n})_{\epsilon_0, \epsilon_1}^{(P)}$$

be an open subset with the property that $g(W) \subset \tilde{N}$ for all $g \in U$. By definition of the C^∞ topology, such a subset does indeed exist.

Claim A.3. *There is an open subset $U' \subset U$ such that for all $g \in U'$, the map $f^{-1} \circ \pi \circ g$ is an element of $\text{Diff}(W)_{\epsilon_0, \epsilon_1}^{(P)}$.*

Proof of Claim. We define $C^\infty(W)_{\epsilon_0, \epsilon_1}^{(P)}$ to be the space of all smooth maps $f : W \rightarrow W$ that satisfy the conditions of Definition 3.1 but may fail to be a diffeomorphism; i.e. smooth maps $f : W \rightarrow W$ that respect corners, collars of length ϵ_0, ϵ_1 , and with $f|_{\partial_1 W} = f_{\beta W} \times Id_p$ for a smooth map $f_{\beta W} : \beta W \rightarrow \beta W$. It is easy to see that

$$\text{Diff}(W)_{\epsilon_0, \epsilon_1}^{(P)} \subset C^\infty(W)_{\epsilon_0, \epsilon_1}^{(P)}$$

is an open subset. We define a map

$$\alpha : \text{Emb}(W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n})_{\epsilon_0, \epsilon_1}^{(P)} \rightarrow C^\infty(W)_{\epsilon_0, \epsilon_1}^{(P)}$$

by the formula $\alpha(g) = f^{-1} \circ \pi \circ g$. One can check that by this formula, $\alpha(g)$ is actually an element of $C^\infty(W)_{\epsilon_0, \epsilon_1}^{(P)}$. It is also clear that α is continuous. Now, $\alpha(f) = Id_W$. Since $\text{Diff}(W)_{\epsilon_0, \epsilon_1}^{(P)}$ is open in $C^\infty(W)_{\epsilon_0, \epsilon_1}^{(P)}$ we can choose a neighborhood V about Id_W in $C^\infty(W)_{\epsilon_0, \epsilon_1}^{(P)}$ such that

$$V \subset \text{Diff}(W)_{\epsilon_0, \epsilon_1}^{(P)}.$$

We now set $U' := \alpha^{-1}(V) \cap U$. This set has the desired properties. \square

Using the same proof, it follows that for all $g \in U'$ and all $\hat{f} \in q^{-1}([f])$, $\hat{f}^{-1} \circ \pi \circ g$ is an element of $\text{Diff}(W)_{\epsilon_0, \epsilon_1}^{(P)}$.

Now let U' be chosen as in Claim A.3. Notice that by the result of Claim A.3, for any $g \in U'$, the restriction $\pi|_{g(W)}$ is injective and a diffeomorphism onto $f(W)$. By how π is defined, this implies that for each $g \in U'$ and $x \in f(W)$, there is a unique geodesic segment contained in \tilde{N} which starts at x with initial velocity orthogonal to $f(W)$ and ends at a point in $g(W)$. We denote such a geodesic by $\Gamma_x^{[g]}$. Notice again that the submanifold $g(W)$

only depends on the class $[g]$ so this curve is well defined. Now for $g \in U'$ and $x \in f(W)$ we define $\lambda([g], x) := \text{dist}(g(W), x)$. Notice that $\Gamma_x^{[g]}(\lambda([g], x))$ is the unique point along the curve segment $\Gamma_x^{[g]}$ that lies in $g(W)$. Also note that

$$\pi(\Gamma_x^{[g]}(\lambda([g], x))) = x$$

for all $g \in U'$ and $x \in f(W)$.

Let $\hat{U}' := q(U')$. We construct a local trivialization

$$\phi : \hat{U}' \times q^{-1}([f]) \longrightarrow q^{-1}(\hat{U}')$$

by setting

$$\phi([g], \tilde{f})(x) = \Gamma_{\tilde{f}(x)}^{[g]}(\lambda([g], \tilde{f}(x)))$$

where $x \in W$, $[g] \in \hat{U}'$ and $\tilde{f} \in q^{-1}([f])$. It follows from Claim A.3 that $\phi([g], \tilde{f})$ is an embedding and it is easy to see that $q(\phi([g], \tilde{f})) = [g]$. Continuity of ϕ follows from the fact $\lambda([g], x) = \text{dist}(g(W), x)$ is a continuous function of $\hat{U}' \times f(W)$ and from the fact that the exponential function is smooth. The inverse to ϕ is defined by setting

$$\phi^{-1}(g) = ([g], \pi \circ g).$$

□

The proof of this theorem can be easily adjusted to show to that

$$q : \text{Emb}(W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n})^{\langle P \rangle} \longrightarrow \frac{\text{Emb}(W, [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n})^{\langle P \rangle}}{\text{Diff}(W)^{\langle P \rangle}}$$

is a locally trivial fibre-bundle. Taking the direct limit as $n \rightarrow \infty$ gives a proof of Theorem 3.3.

Let W be a d -dimensional manifold with corners modeled on $[0, \infty)^2 \times \mathbb{R}^{m-2}$, in other words a P -manifold with $P = \star$. Let

$$\text{Emb}(W, [0, \infty)^2 \times \mathbb{R}^{m-2})_{\epsilon_0, \epsilon_1}^{\langle \star \rangle}$$

and

$$\text{Emb}(\partial_0 W, \{0\} \times [0, \infty) \times \mathbb{R}^{m-2})_{\epsilon_1}^{\langle \star \rangle}$$

be spaces of neat embeddings which respect collars of width ϵ_0 and ϵ_1 .

Theorem A.4. *The restriction map*

$$r : \text{Emb}(W, [0, \infty)^2 \times \mathbb{R}^{m-2})_{\epsilon_0, \epsilon_1}^{\langle \star \rangle} \longrightarrow \text{Emb}(\partial_0 W, \{0\} \times [0, \infty) \times \mathbb{R}^{m-2})_{\epsilon_1}^{\langle \star \rangle}$$

is a locally trivial fibre-bundle.

Proof. Let $f \in \text{Emb}(\partial_0 W, \{0\} \times [0, \infty) \times \mathbb{R}^{m-2})_{\epsilon_1}^{\langle \star \rangle}$. To prove the theorem it will suffice to construct a map

$$(73) \quad \eta : U \longrightarrow \text{Diff}([0, \infty)^2 \times \mathbb{R}^{m-2})_{\epsilon_0, \epsilon_1}^{\langle \star \rangle}$$

where U is a suitably chosen neighborhood of f in $\text{Emb}(\partial_0 W, \{0\} \times [0, \infty) \times \mathbb{R}^{m-2})_{\epsilon_1}^{(*)}$, such that $\eta(g) \circ f = g$ for all $g \in U$ and $\eta(f) = \text{Id}$. We note that in (73), by $\text{Diff}([0, \infty)^2 \times \mathbb{R}^{m-2})_{\epsilon_0, \epsilon_1}^{(*)}$ we mean the space of diffeomorphisms with compact supports. A diffeomorphism with compact support is a diffeomorphism that is the identity outside of some compact subspace. With such a map η constructed we obtain a local trivialization

$$\phi : U \times r^{-1}(f) \longrightarrow r^{-1}(U)$$

defined by the formula

$$\phi(g, \hat{f}) = \eta(g)(\hat{f})$$

where $g \in U$ and $\hat{f} \in r^{-1}(f)$. We see that

$$r(\phi(g, \hat{f})) = r(\eta(g) \circ \hat{f}) = \eta(g) \circ r(\hat{f}) = \eta(g) \circ f = g.$$

The inverse to ϕ is defined by the formula,

$$\phi^{-1}(\hat{g}) = (g, \eta(g)^{-1} \circ \hat{g})$$

where $\hat{g} \in r^{-1}(U)$ and $r(\hat{g}) = g$. Since

$$\eta(g) \circ f = g \implies f = \eta(g)^{-1} \circ g \quad \text{we have,}$$

$$r(\eta(g)^{-1} \circ \hat{g}) = \eta^{-1}(g) \circ g = f$$

which implies that $\phi^{-1}(\hat{g})$ is actually in $U \times r^{-1}(f)$. It is easy to check that this map is actually the inverse to ϕ .

Claim A.5. *There exists a map η as in (73) such that $\eta(g) \circ f = g$ for all $g \in U$ and $\eta(f) = \text{Id}$.*

Proof of Claim. Let f be as above. Let $N \rightarrow f(\partial_0 W)$ be the normal bundle of $f(\partial_0 W)$ in $\{0\} \times [0, \infty) \times \mathbb{R}^{m-2}$. Let $\delta > 0$ be a real number such that the restriction of

$$\exp : N \rightarrow \{0\} \times [0, \infty) \times \mathbb{R}^{m-2}$$

to the subspace $N_\delta := \{(x, v) \in N \mid |v| < \delta\}$ is an embedding. We then define

$$\tilde{N}_\delta := \exp(N_\delta).$$

\tilde{N}_δ is a tubular neighborhood about $f(\partial_0 W)$ in $\{0\} \times [0, \infty) \times \mathbb{R}^{m-2}$ with the property that if $\text{dist}(x, f(\partial_0 W)) < \delta$ then $x \in \tilde{N}_\delta$. Clearly if $\delta' < \delta$ then $\tilde{N}_{\delta'} \subset \tilde{N}_\delta$. Let $\pi : \tilde{N}_\delta \rightarrow f(\partial_0 W)$ be the projection map. Notice the factorization,

$$N|_{f(\partial_{0,1} W) \times [0, \epsilon_1]} = N_{f(\partial_{0,1} W)} \times \epsilon^1$$

where $N_{f(\partial_{0,1} W)}$ is the normal bundle for $f(\partial_{0,1} W)$ in $\{0\} \times \{0\} \times \mathbb{R}^{m-2}$. By a similar argument as made in the proof of Claim A.2, we see that for $(s, x) \in \{0\} \times [0, \epsilon_1) \times \mathbb{R}^{m-2}$ with $s \in [0, \epsilon_1)$ and $x \in \mathbb{R}^{m-2}$, we have $\pi(s, x) = (s, \pi^{\partial_{0,1}}(x))$ where $\pi^{\partial_{0,1}}$ is the projection map for a tubular neighborhood of $f(\partial_{0,1} W)$ in $\{0\} \times \{0\} \times \mathbb{R}^{m-2}$ corresponding to the normal bundle $N_{f(\partial_{0,1} W)}$.

Now let U' be an open neighborhood in $\text{Emb}(\partial_0 M, \{0\} \times [0, \infty) \times \mathbb{R}^{m-2})_{\epsilon_1}^{(\partial)}$ with the property that $g(\partial_0 W) \subseteq \tilde{N}_\delta$ for all $g \in U'$. Let

$$\lambda : [0, \infty) \longrightarrow [0, 1]$$

be a smooth non-increasing function such that $\lambda(t) = 1$ for $t \leq \frac{1}{4}\delta$ and $\lambda(t) = 0$ for $t \geq \frac{1}{2}\delta$. Let

$$\zeta : [0, \infty) \longrightarrow [0, \infty)$$

be a smooth function with $\zeta(t) = 0$ for $t \leq \epsilon_0$, $\zeta(t) \geq \delta$ for $t \geq 2\epsilon_0$ such that ζ is strictly-increasing on (ϵ_0, ∞) . Let

$$p : \mathbb{R}^m \longrightarrow [0, \infty)^2 \times \mathbb{R}^{m-2}$$

be the standard linear retraction. We define a function

$$\eta' : U' \longrightarrow C^\infty([0, \infty)^2 \times \mathbb{R}^{m-2}, [0, \infty)^2 \times \mathbb{R}^{m-2})_{\epsilon_0, \epsilon_1}^{(\star)}$$

(where $C^\infty([0, \infty)^2 \times \mathbb{R}^{m-2}, [0, \infty)^2 \times \mathbb{R}^{m-2})_{\epsilon_0, \epsilon_1}^{(\star)}$ is understood to consist of functions with compact support) by setting $\eta(g)(s_0, s_1, x) = (s_0, s_1, x)$ if $(0, s_1, x) \notin \tilde{N}_\delta$ and by setting

$$\eta'(g)(s_0, s_1, x) = p\left((s_0, s_1, x) + \lambda\{|\zeta(s_0), \pi(s_1, x) - (s_1, x)|\} \cdot [(0, gf^{-1}\pi(s_1, x) - \pi(s_1, x))]\right)$$

when $(0, s_1, x) \in \tilde{N}_\delta$. In the above formula, $|\cdot|$ is the Euclidean norm and the addition and subtraction symbols are the standard vector space addition and subtraction in \mathbb{R}^m . One can verify that for each $g \in U'$, we have

$$\eta(g) \in C^\infty([0, \infty)^2 \times \mathbb{R}^{m-2}, [0, \infty)^2 \times \mathbb{R}^{m-2})_{\epsilon_0, \epsilon_1}^{(\star)} \quad \text{and that}$$

$$\eta(g) \circ f = g \quad \text{and} \quad \eta(f) = Id.$$

Now since $\text{Diff}([0, \infty)^2 \times \mathbb{R}^{m-2})_{\epsilon_0, \epsilon_1}^{(\star)}$ is open in $C^\infty([0, \infty)^2 \times \mathbb{R}^{m-2}, [0, \infty)^2 \times \mathbb{R}^{m-2})_{\epsilon_0, \epsilon_1}^{(\star)}$ and $\eta(f) = Id$ is a diffeomorphism, we can choose an open neighborhood U about f contained in U' such that

$$\eta(g) \in \text{Diff}([0, \infty)^2 \times \mathbb{R}^{m-2})_{\epsilon_0, \epsilon_1}^{(\star)}$$

for all $g \in U$. We then define η to simply be the restriction of η' to U . This gives the proof. \square

\square

Taking the direct limits as $\epsilon_0, \epsilon_1 \rightarrow \infty$ and $m \rightarrow \infty$ gives the proof of Lemma 3.2.

APPENDIX B.

In this section we prove several results which together imply Proposition 7.10. This is very similar to the proof of [11, 3.2.5].

Proposition B.1. *The spaces $|\hat{\mathbf{D}}_{d+1}^{P,m}|$ and $\Omega^{\infty-1}\mathbf{MT}(d+1)_{\langle P,\partial,m \rangle}$ both have the structure of topological monoids up to homotopy.*

Proof. The monoid (up to homotopy) structure on $|\hat{\mathbf{D}}_{d+1}^{P,m}|$ is defined as follows: Let $\hat{\mathbf{D}}_{d+1}^{P,m} \bar{\times} \hat{\mathbf{D}}_{d+1}^{P,m}$ be the sheaf defined by letting $\hat{\mathbf{D}}_{d+1}^{P,m} \bar{\times} \hat{\mathbf{D}}_{d+1}^{P,m}(X)$ consist of all pairs

$$((W^1; \pi^1, f^1, e^1), (W^2; \pi^2, f^2, e^2)) \in \hat{\mathbf{D}}_{d+1}^{P,m}(X) \times \hat{\mathbf{D}}_{d+1}^{P,m}(X)$$

such that the image of e^1 and e^2 are disjoint in $X \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n}$. There is a natural map

$$(74) \quad \mu : \hat{\mathbf{D}}_{d+1}^{P,m} \bar{\times} \hat{\mathbf{D}}_{d+1}^{P,m}(X) \longrightarrow \hat{\mathbf{D}}_{d+1}^{P,m}(X)$$

defined by sending a pair $((W^1; \pi^1, f^1, e^1), (W^2; \pi^2, f^2, e^2))$ to the element

$$(W^1 \sqcup W^2; \pi^1 \sqcup \pi^2, f^1 \sqcup f^2, e^1 \sqcup e^2).$$

This map yields a partially defined product on $\hat{\mathbf{D}}_{d+1}^{P,m}(X)$ which is clearly associative and commutative. The identity element is given by $(\emptyset; \emptyset, \emptyset, \emptyset)$ (here the manifold is \emptyset and the maps are the empty maps). It is easy to see that the inclusion map

$$j : \hat{\mathbf{D}}_{d+1}^{P,m} \bar{\times} \hat{\mathbf{D}}_{d+1}^{P,m} \longrightarrow \hat{\mathbf{D}}_{d+1}^{P,m} \times \hat{\mathbf{D}}_{d+1}^{P,m}$$

is a weak equivalence of sheaves. Roughly, given

$$((W^1; \pi^1, f^1, e^1), (W^2; \pi^2, f^2, e^2)) \in \hat{\mathbf{D}}_{d+1}^{P,m}(X) \times \hat{\mathbf{D}}_{d+1}^{P,m}(X)$$

such that the images of e_1 and e_2 intersect, after increasing the dimension of the ambient space, one can easily find an isotopy of embeddings which pulls W_1 away from W_2 .

Letting $|k|$ be a pseudo-inverse for $|j|$, the product described above yields a homotopy monoid structure on the representing space $|\hat{\mathbf{D}}_{d+1}^{P,m}|$ with product given by

$$(75) \quad |\hat{\mathbf{D}}_{d+1}^{P,m}| \times |\hat{\mathbf{D}}_{d+1}^{P,m}| \xrightarrow{\sim} |\hat{\mathbf{D}}_{d+1}^{P,m} \times \hat{\mathbf{D}}_{d+1}^{P,m}| \xrightarrow{|k|} |\hat{\mathbf{D}}_{d+1}^{P,m} \bar{\times} \hat{\mathbf{D}}_{d+1}^{P,m}| \xrightarrow{|\mu|} |\hat{\mathbf{D}}_{d+1}^{P,m}|$$

where the left-most map is some choice of homotopy equivalence. The empty-set element in $\hat{\mathbf{D}}_{d+1}^{P,m}(\star)$ (which induces the empty-set element in $\hat{\mathbf{D}}_{d+1}^{P,m}(X)$ for any X by pulling back over the constant map) determines an element $e \in |\hat{\mathbf{D}}_{d+1}^{P,m}|$. This is easy to see by examining the construction of $|\hat{\mathbf{D}}_{d+1}^{P,m}|$ as the geometric realization of the simplicial set $(l \mapsto \hat{\mathbf{D}}_{d+1}^{P,m}(\Delta^l))$. From the fact that the empty set is the identity for the partially defined product in (74) it follows that e is the identity (up to homotopy) for the product defined in (75). Associativity also follows from associativity of (74). The monoid structure on $\Omega^{\infty-1}\mathbf{MT}(d+1)_{\langle P,\partial,m \rangle}$ is defined in the usual way via addition of loops. \square

The above proposition implies that both $\pi_0(|\hat{\mathbf{D}}_{d+1}^{P,m}|)$ and $\pi_0(\Omega^{\infty-1}\mathbf{MT}(d+1)_{\langle P,\partial,m \rangle})$ are actual monoids. Since $\Omega^{\infty-1}\mathbf{MT}(d+1)_{\langle P,\partial,m \rangle}$ is an infinite loop space, $\pi_0(\Omega^{\infty-1}\mathbf{MT}(d+1)_{\langle P,\partial,m \rangle})$ is actually a group since all loops have inverses up to homotopy.

Proposition B.2. *The map*

$$|H^m| \circ |\hat{T}^m| : |\hat{\mathbf{D}}_{d+1}^{P,m}| \longrightarrow \Omega^{\infty-1}\mathbf{MT}(d+1)_{\langle P,\partial,m \rangle}$$

induces a monoid homomorphism

$$\pi_0(|\hat{\mathbf{D}}_{d+1}^{P,m}|) \longrightarrow \pi_0(\Omega^{\infty-1}\mathbf{MT}(d+1)_{\langle P,\partial,m \rangle}).$$

By lemma 7.2, this homomorphism is an isomorphism of monoids. Thus $\pi_0(|\hat{\mathbf{D}}_{d+1}^{P,m}|)$ is actually a group.

Proof. This can be proven by examining the map

$$\hat{T}^m : \hat{\mathbf{D}}_{d+1}^{P,m}(\star) \longrightarrow \mathcal{Z}_{d+1}^{P,m}(\star)$$

Let $((W_1; \pi_1, f_1, e_1), (W_2; \pi_2, f_2, e_2))$ represent an element of $\mathbf{D}_{d+1}^{\hat{P},m} \times \bar{\mathbf{D}}_{d+1}^{P,m}[\star]$. We may assume that W_1 and W_2 are P -submanifolds of $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n}$ and thus come from elements of $\hat{\mathbf{D}}_{d+1,n}^{P,m}$. We examine the image of the class representing $(W_1 \sqcup W_2; \pi_1 \sqcup \pi_2, f_1 \sqcup f_2, e_1 \sqcup e_2)$ under \hat{T}^m . Notice that since the images of e_1 and e_2 are disjoint, the Thom-collapse map induced by the embedding $e_1 \sqcup e_2 : N_{W_1} \sqcup N_{W_2} \hookrightarrow \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n}$ has the factorization

$$(76) \quad \begin{array}{ccccc} \mathbb{R} \times D^{d+n} & \xrightarrow{\quad} & \mathbb{R} \times (D^{d+n} \vee D^{d+n}) & \xrightarrow{\quad} & \mathrm{Th}(N_{W_1}) \vee \mathrm{Th}(N_{W_2}) \\ \uparrow & & \uparrow & & \uparrow \\ \mathbb{R} \times S^{d+n-1-p-m} \wedge S^{p+m} & \xrightarrow{\quad} & \mathbb{R} \times (S^{d+n-1-p-m} \vee S^{d+n-1-p-m}) \wedge S^{p+m} & \xrightarrow{\quad} & (\mathrm{Th}(\beta W_1) \vee \mathrm{Th}(\beta W_2)) \wedge S^{p+m} \end{array}$$

where the left-most maps in the top and bottom rows of the diagram are the standard equatorial pinch-maps. To be explicit, viewing $D^{d+n} = (\mathbb{R}_+ \times \mathbb{R}^{d-1+n})^c$ (the one-point compactification of $\mathbb{R}_+ \times \mathbb{R}^{d-1+n}$), this pinch map which we denote by

$$p : (\mathbb{R}_+ \times \mathbb{R}^{d-1+n})^c \rightarrow (\mathbb{R}_+ \times \mathbb{R}^{d-1+n})^c \vee (\mathbb{R}_+ \times \mathbb{R}^{d-1+n})^c$$

is defined by collapsing the hyperplane

$$\{(x_0, (x_1, \dots, x_{d+n-1})) \in \mathbb{R}_+ \times \mathbb{R}^{d-1+n} \mid x_1 = 0\}.$$

The restriction of p to $(\{0\} \times \mathbb{R}^{d+n})^c$ can then be seen to factor as a smash product

$$p_0 \wedge Id : (\{0\} \times \mathbb{R}^{d-1+n-p-m})^c \wedge (\mathbb{R}^{p+m})^c \rightarrow [(\{0\} \times \mathbb{R}^{d-1+n-p-m})^c \vee (\{0\} \times \mathbb{R}^{d-1+n-p-m})^c] \wedge (\mathbb{R}^{p+m})^c$$

where p_0 is defined by collapsing the hyperplane

$$\{(x_1, \dots, x_{d+n-1-p-m}) \in \mathbb{R}^{d-1+n-p-m} \mid x_1 = 0\},$$

thus yielding commutativity in the left square of (76). The Thom-Pontryagin map for $W_1 \sqcup W_2$ then takes the form of a composition of the pinch-map p , followed by the map given by

$$\begin{array}{ccc} \mathbb{R} \times (D^{d+n} \vee D^{d+n}) & \xrightarrow{\quad\quad\quad} & \mathrm{Th}(U_{d+1,n}^\perp) \vee \mathrm{Th}(U_{d+1,n}^\perp) \\ \uparrow & & \uparrow \\ \mathbb{R} \times (S^{d+n-1-p-m} \vee S^{d+n-1-p-m}) \wedge S^{p+m} & \xrightarrow{\quad\quad\quad} & (\mathrm{Th}(U_{d-p,n-m}^\perp) \vee \mathrm{Th}(U_{d-p,n-m}^\perp)) \wedge S^{p+m}, \end{array}$$

$\tilde{j}_d^n \circ \tau_{P,m}^n \vee \tilde{j}_d^n \circ \tau_{P,m}^n$

and then followed by the standard fold map

$$\begin{array}{ccc} \mathrm{Th}(U_{d+1,n}^\perp) \vee \mathrm{Th}(U_{d+1,n}^\perp) & \xrightarrow{\quad\quad\quad} & \mathrm{Th}(U_{d+1,n}^\perp) \\ \uparrow & & \uparrow \\ \mathrm{Th}(U_{d-p,n-m}^\perp) \vee \mathrm{Th}(U_{d-p,n-m}^\perp) & \xrightarrow{\quad\quad\quad} & (\mathrm{Th}(U_{d-p,n-m}^\perp)) \wedge S^{p+m}. \end{array}$$

$\tilde{j}_d^n \circ \tau_{P,m}^n \vee \tilde{j}_d^n \circ \tau_{P,m}^n$

This induces a map $\mathbb{R} \rightarrow \Omega^{d+n} \mathrm{Cofibre}(\tilde{j}_d^n \circ \tau_{P,m}^n)$ which is clearly the sum of the Thom-Pontryagin maps for W_1 and W_2 induced by the vertical tubular neighborhoods for their normal bundles. Passing to the direct limit as $n \rightarrow \infty$ we see that \hat{T}^m respects the monoid structures. \square

We can now finish the proof of Proposition 7.10. Let $e \in |\hat{\mathbf{D}}_{d+1}^{P,m}|$ be the identity element for the monoid structure. The monoid structure implies that for any $k > 0$,

$$[(S^k, s_0); (|\hat{\mathbf{D}}_{d+1}^{P,m}|, e)] \cong [S^k, |\hat{\mathbf{D}}_{d+1}^{P,m}|_e]$$

where $|\hat{\mathbf{D}}_{d+1}^{P,m}|_e$ denotes the path component of e . This isomorphism of the set of classes of based-maps with the set of classes of un-based-maps follows from the fact that π_1 always acts trivially on the path component of the identity of such spaces with a homotopy-monoid structure, see [8, 4A.3]. A similar statement holds for the component of the identity element in $\Omega^{\infty-1} \mathbf{MT}(d+1)_{\langle P, \partial, m \rangle}$ as well. Now, the monoid structure yields a group action of $\pi_0(|\hat{\mathbf{D}}_{d+1}^{P,m}|)$ on $[S^k, |\hat{\mathbf{D}}_{d+1}^{P,m}|]$. For any two path components $|\hat{\mathbf{D}}_{d+1}^{P,m}|_{x_0}, |\hat{\mathbf{D}}_{d+1}^{P,m}|_{x_1}$, this group action of π_0 yields an isomorphism,

$$[S^k, |\hat{\mathbf{D}}_{d+1}^{P,m}|_{x_0}] \cong [S^k, |\hat{\mathbf{D}}_{d+1}^{P,m}|_{x_1}].$$

The same isomorphism holds for the different path componenets of $\Omega^{\infty-1}\mathbf{MT}(d+1)_{\langle P, \partial, m \rangle}$ as well. These two observations together imply the isomorphisms

$$\pi_k(|\hat{\mathbf{D}}_{d+1}^{P,m}|) \cong \frac{[S^k, |\hat{\mathbf{D}}_{d+1}^{P,m}|]}{\pi_0(|\hat{\mathbf{D}}_{d+1}^{P,m}|)} \quad \text{and}$$

$$\pi_k(\Omega^{\infty-1}\mathbf{MT}(d+1)_{\langle P, \partial, m \rangle}) \cong \frac{[S^k, \Omega^{\infty-1}\mathbf{MT}(d+1)_{\langle P, \partial, m \rangle}]}{\pi_0(\Omega^{\infty-1}\mathbf{MT}(d+1)_{\langle P, \partial, m \rangle})}$$

for any choice of base point. From this identification of the homotopy groups, the set isomorphism in Lemma 32 implies the isomorphism of homotopy groups.

APPENDIX C.

In this section we prove two Lemmas that pertain to embeddings of P -manifolds. The first lemma is used to prove Lemma 5.3. The second is used in the proof of 7.3.

Lemma C.1. *Let $X \in \mathcal{Ob}(\mathcal{X})$ and $(W; \pi, f) \in \mathbf{D}_{d+1,n}^{P,m}(X)$. Assume that $X \subset \mathbb{R}^l$ for some l . Now suppose that $n > 2 \cdot \dim(X) + d + 3 - p - m$. Then, the element $(W; \pi, f)$ is concordant to an element that lies in the image of the inclusion $\mathbf{D}_{d+1,n-1}^{P,m}(X) \longrightarrow \mathbf{D}_{d+1,n}^{P,m}(X)$.*

Proof. We use a technique here very similar to the one used in the proof of the *Whitney Embedding Theorem* given in [9, 3.5]. In order to prove this lemma, we seek a vector

$$v \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n-p-m} \times \mathbb{R}^{p+m}$$

which is orthogonal to the subspace $\mathbb{R} \times \mathbb{R}_+ \times \{0\} \times \mathbb{R}^{p+m} \subset \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n-p-m} \times \mathbb{R}^{p+m}$ such that the map

$$e_v : W \longrightarrow X \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n-p-m} \times \mathbb{R}^{p+m}$$

given by

$$(77) \quad w \longmapsto (Id_X(w), proj_{v^\perp}(w)) \longmapsto i((Id_X(w), proj_{v^\perp}(w)))$$

is an embedding, where $i : X \times v^\perp \hookrightarrow X \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n-p-m} \times \mathbb{R}^{p+m}$ the inclusion. Assuming that such a vector v exists, by the fact that v was chosen to be orthogonal to $\mathbb{R} \times \mathbb{R}_+ \times \{0\} \times \mathbb{R}^{p+m}$, the image $e(W)$ will still be a P -manifold with boundary embedded in the correct way specified in the definition of $\mathbf{D}_{d+1,n}^{P,m}$. Furthermore the restriction of the projection onto X will still be a submersion and projection onto $X \times \mathbb{R}$ still proper thus yielding a new element $(e(W); \pi', f') \in \mathbf{D}_{d+1,n}^{P,m}(X)$. It is easy to check that the embedding e is isotopic to the original inclusion of W and thus $(e(W); \pi', f')$ is concordant to $(W; \pi, f)$ in $\mathbf{D}_{d+1,n}^{P,m}(X)$. Notice that $e(W) \subset X \times v^\perp$ where v^\perp is a codimension 1 half-vector subspace of $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n-p-m} \times \mathbb{R}^{p+m}$. Upon applying an appropriate rotation of

$$\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n-p-m} \times \mathbb{R}^{p+m},$$

we see that $(W; \pi, f)$ is concordant to an element in the image of

$$\mathbf{D}_{d+1,n-1}^{P,m}(X) \longrightarrow \mathbf{D}_{d+1,n}^{P,m}(X).$$

Recall that we are assuming that $X \subset \mathbb{R}^l$. It will suffice to find a vector v as above such that the map e_v given by (77) is a proper, injective, immersion. In order for e_v to be injective, it will sufficient to for v to not be parallel to any secant line in $\mathbb{R}^l \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n-p-m} \times \mathbb{R}^{p+m}$ connecting two points of W . The map e_v fails to be injective precisely when v is parallel to some secant line through W .

Consider the map

$$g : W \times W \setminus \Delta \longrightarrow S^{d-1+n-p-m-1}$$

given by

$$g(x, y) = \frac{\text{proj}(x - y)}{\|\text{proj}(x - y)\|}$$

where

$$\text{proj} : \mathbb{R}^l \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n-p-m} \times \mathbb{R}^{p+m} \longrightarrow \mathbb{R}^{d-1+n-p-m}$$

is the projection. By assumption

$$\dim(W \times W \setminus \Delta) = 2(\dim(X) + d + 1) < \dim(S^{d-1+n-p-m-1}) = d - 1 + n - p - m - 1.$$

Since g is a smooth map, the dimensionality implies that the image of g is of measure zero. Therefore we can choose a vector $v' \in S^{d-1+n-p-m-1} \setminus g(W \times W \setminus \Delta)$. We then set v to be the image of v' under the embedding of $S^{d-1+n-p-m-1} \hookrightarrow \mathbb{R}^l \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n-p-m} \times \mathbb{R}^{p+m}$ via zero-section. It is clear that v is not parallel to any secant line through W and that v is orthogonal to $\mathbb{R}^l \times \mathbb{R} \times \mathbb{R}_+ \times \{0\} \times \mathbb{R}^{p+m}$.

Now for v to be such that e_v is an immersion, it will suffice for v to not be parallel to any tangent vector of W . e_v fails to be an immersion precisely when v is parallel to some tangent vector to W . For what follows, view the tangent space TW as a subspace of

$$W \times \mathbb{R}^l \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n-p-m} \times \mathbb{R}^{p+m}.$$

We then define

$$q : TW \longrightarrow S^{d-1+n-p-m-1} \quad \text{by} \quad q(v) = \frac{\text{proj}(\pi(v))}{\|\text{proj}(\pi(v))\|},$$

where

$$\pi : W \times \mathbb{R}^l \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n-p-m} \times \mathbb{R}^{p+m} \longrightarrow \mathbb{R}^l \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n-p-m} \times \mathbb{R}^{p+m}$$

is the projection and proj is the same projection used above. Like in the previous case, $\dim(TW) < \dim(S^{d-1+n-p-m-1})$ and since q is a smooth map, $q(TW)$ has measure zero in $S^{d-1+n-p-m-1}$. We then may choose v' in the compliment of $q(TW) \cup g(W \times W \setminus \Delta)$. By embedding $S^{d-1+n-p-m-1}$ into $\mathbb{R}^l \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n-p-m} \times \mathbb{R}^{p+m}$ via zero-section, we obtain a vector v orthogonal to $\mathbb{R}^l \times \mathbb{R} \times \mathbb{R}_+ \times \{0\} \times \mathbb{R}^{p+m}$ such that e_v is an injective immersion.

Now, by definition of $\mathbf{D}_{d+1,n}^{P,m}(X)$, the map $(\pi, f) : W \longrightarrow X \times \mathbb{R}$ given by projection is proper. By how v was chosen, the subspace v^\perp must contain the subspace

$$\mathbb{R} \times \{0\} \times \{0\} \times \{0\} \subset \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d-1+n-p-m} \times \mathbb{R}^{p+m}.$$

This implies that the projection (π, f) factors through the projection e from (77) and thus e is a proper map as well as being an injective immersion for the right choice of v . This proves the lemma. \square

A similar lemma can also be proven for the sheaf $\hat{\mathbf{D}}_{d+1,n}^{P,m}$. Lemma C.1 implies that

$$\pi_k(|\mathbf{D}_{d+1,n-1}^{P,m}|) \longrightarrow \pi_k(|\mathbf{D}_{d+1,n}^{P,m}|)$$

is surjective whenever $n > d + 3 + 2k + p + m$. If $n > d + 5 + 2k + p + m$ then

$$\pi_k(|\mathbf{D}_{d+1,n-1}^{P,m}|) \longrightarrow \pi_k(|\mathbf{D}_{d+1,n}^{P,m}|)$$

is injective as well. This can be seen by applying the above Lemma to show that any concordance, an element of $\mathbf{D}_{d+1,n}^{P,m}(S^k \times \mathbb{R})$, is concordant to an element of $\mathbf{D}_{d+1,n-1}^{P,m}(S^k \times \mathbb{R})$. This proves Lemma 5.3.

Lemma C.2 (Whitney Approximation For P -Manifolds). *Fix an embedding $i_P : P \hookrightarrow \mathbb{R}^{p+m}$. Let W be a compact P manifold of dimension d with $\partial_0 W = \emptyset$. Let*

$$j : W \longrightarrow \mathbb{R}_+ \times \mathbb{R}^{n-p-m} \times \mathbb{R}^{p+m}$$

be an element of $C^\infty(W, \mathbb{R}_+ \times \mathbb{R}^{n-p-m} \times \mathbb{R}^{p+m})^{(P,m)}$, see (8). If $n > 2d + p + m$, then j can be approximated by an element of $\text{Emb}(W, \mathbb{R}_+ \times \mathbb{R}^{n-p-m} \times \mathbb{R}^{p+m})^{(P,m)}$.

Proof. Let j as in the statement of the Lemma be given. We may assume that ∂W is given a collar $\partial W \times [0, 1] \subset W$ and that j is a smooth map with the property that if $(t, w, p) \in [0, 1] \times \beta W \times P = [0, 1] \times \partial W$ then

$$j(t, w, p) = (t, j_{\beta W}, i_P(p)) \in [0, 1] \times \mathbb{R}^{n-p-m} \times \mathbb{R}^{p+m}$$

where $j_{\beta W} : \beta W \longrightarrow \mathbb{R}^{n-p-m}$ is a smooth map. Choose a P -manifold embedding (an element of $\text{Emb}(W, \mathbb{R}_+ \times \mathbb{R}^{n-p-m} \times \mathbb{R}^{p+m})^{(P,m)}$)

$$e : W \hookrightarrow \mathbb{R}_+ \times \mathbb{R}^{r-p-m} \times \mathbb{R}^{p+m}$$

which respects the collar $[0, 1] \times \partial W$. For $(t, w, p) \in [0, 1] \times \beta W \times P$ we have

$$e(t, w, p) = (t, e_{\beta W}, i_P(p)) \in \mathbb{R}_+ \times \mathbb{R}^{r-p-m} \times \mathbb{R}^{p+m}$$

for some embedding $e_{\beta W} : \beta W \hookrightarrow \mathbb{R}^{r-p-m}$. We follow this embedding with the inclusion

$$\mathbb{R}_+ \times \mathbb{R}^{r-p-m} \times \mathbb{R}^{p+m} \hookrightarrow \mathbb{R} \times \mathbb{R}^{r-p-m} \times \mathbb{R}^{p+m}.$$

and denote (with abuse of notation) by $e : W \hookrightarrow \mathbb{R} \times \mathbb{R}^{r-p-m} \times \mathbb{R}^{p+m}$ the resulting composition. Now consider the map

$$j \times e : W \longrightarrow (\mathbb{R}_+ \times \mathbb{R}^{n-p-m} \times \mathbb{R}^{p+m}) \times (\mathbb{R} \times \mathbb{R}^{r-p-m} \times \mathbb{R}^{p+m}).$$

This map is an embedding of W as a manifold with boundary, however it is not an element of $\text{Emb}(W, \mathbb{R}_+ \times \mathbb{R}^{n-p-m} \times \mathbb{R}^{p+m})^{\langle P, m \rangle}$. We will need to alter this map. Let

$$\phi : \mathbb{R} \longrightarrow [0, 1]$$

be a smooth map with $\phi(t) = 0$ when $t < \frac{1}{4}$ and with $\phi(t) = 1$ when $t > \frac{3}{4}$ such that ϕ is strictly increasing on $(\frac{1}{4}, \frac{3}{4})$. We now define

$$\Phi : \mathbb{R} \times \mathbb{R}^{r-p-m} \times \mathbb{R}^{p+m} \longrightarrow \mathbb{R} \times \mathbb{R}^{r-p-m} \times \mathbb{R}^{p+m}$$

by the formula

$$\Phi(t, x, z) = (t, x, \phi(t) \cdot z)$$

for $(t, x, z) \in \mathbb{R} \times \mathbb{R}^{r-p-m} \times \mathbb{R}^{p+m}$. Now consider the map

$$j \times (\Phi \circ e) : W \longrightarrow (\mathbb{R}_+ \times \mathbb{R}^{n-p-m} \times \mathbb{R}^{p+m}) \times (\mathbb{R} \times \mathbb{R}^{r-p-m} \times \mathbb{R}^{p+m}).$$

For $(t, w, p) \in [0, \frac{1}{4}] \times \beta W \times P$, by the definition of Φ we have

$$(j \times (\Phi \circ e))(t, w, p) = ((t, j_{\beta W}(w), i_P(p)), (t, e_{\beta W}(b), 0)), \text{ where}$$

$$((t, j_{\beta W}(w), i_P(p)), (t, e_{\beta W}(b), 0)) \in (\mathbb{R}_+ \times \mathbb{R}^{n-p-m} \times \mathbb{R}^{p+m}) \times (\mathbb{R} \times \mathbb{R}^{r-p-m} \times \mathbb{R}^{p+m}).$$

Since $e_{\beta W}$ is an embedding, it follows that up to permutation of factors, $j \times (\Phi \circ e)$ is a P -manifold embedding on the collar $[0, \frac{1}{4}] \times \partial W$. It can also be checked by examining the definition of Φ that $j \times (\Phi \circ e)$ is an embedding on all of W and so $j \times (\Phi \circ e)$ is an element of $\text{Emb}(W, \mathbb{R}_+ \times \mathbb{R}^{n-p-m} \times \mathbb{R}^{p+m})^{\langle P, m \rangle}$. We now must alter this embedding to one that is arbitrarily close (in the C^k metric for any k) to the original map j . Notice that the map j is equal to the composition $\pi \circ (j \times (\Phi \circ e))$ where

$$\pi : (\mathbb{R}_+ \times \mathbb{R}^{n-p-m} \times \mathbb{R}^{p+m}) \times (\mathbb{R} \times \mathbb{R}^{r-p-m} \times \mathbb{R}^{p+m}) \longrightarrow \mathbb{R}_+ \times \mathbb{R}^{n-p-m} \times \mathbb{R}^{p+m}$$

is the projection. It will suffice then to find an appropriate $(n+1)$ -dimensional subspace

$$V \subset (\mathbb{R}_+ \times \mathbb{R}^{n-p-m} \times \mathbb{R}^{p+m}) \times (\mathbb{R} \times \mathbb{R}^{r-p-m} \times \mathbb{R}^{p+m})$$

with $(\mathbb{R}_+ \times \{0\} \times \mathbb{R}^{p+m}) \times (\{0\} \times \{0\} \times \{0\}) \subset V$ such that the map given by

$$\text{proj}_V \circ (j \times (\Phi \circ e))$$

is an embedding with proj_V arbitrarily close, within ϵ -distance in the C^∞ norm for any chosen $\epsilon > 0$, to the projection π . For n sufficiently large, the proof of existence of such a projection proceeds exactly as in the proof of the previous lemma. The sufficiency of the estimate

$$n > 2d + p + m$$

is easy to check using the technique in the proof of the previous lemma. \square

REFERENCES

- [1] N Baas, On Bordism Theory of Manifolds with Singularities, Math. Scan. 33 (1973) 279-302
- [2] N Baas, A Note on Cobordism Categories, Preprint, NTNU, 2009
- [3] E. Binz, H. R. Fischer, The Manifold of Embeddings of a Closed Manifold. Differential Geometric Methods in Mathematical Physics Lecture Notes in Physics Volume 139, (1981), pp 310-325
- [4] B. Botvinnik, Manifolds with Singularities and the Adams-Novikov Spectral Sequence. Cambridge University Press (1992)
- [5] Y. Eliashberg, N. Mishachev, Introduction to the h-Principle. AMS (2002)
- [6] S. Galatius, Ib Madsen, U. Tillman, M. Weiss, The Homotopy Type of a Cobordism Category, Acta Math., 202 (2009), 195-239
- [7] J. Genauer, Cobordism Categories of Manifolds with Corners, arXiv:0810.0581 (2008)
- [8] A. Hatcher, Algebraic Topology. Cambridge University Press (2001)
- [9] M. Hirsch, Differential Topology. Springer-Verlag New York Inc. (1976)
- [10] E. Lima, On The Local Triviality of the Restriction Map for Embeddings. Comentarîi Mathematici Helvetici (1963)
- [11] I. Madsen, M. Weiss, The Stable Moduli Space of Riemann Surfaces: Mumford's Conjecture. Annals of Mathematics (2) 165 (2007), no. 3, 843-941
- [12] A. Phillips, Submersions of Open Manifolds. Topology. 6:171-206 (1967)

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